Solutions to Assignment #2

1. Let \( A, B \) and \( C \) be subsets of a sample space \( C \). Prove the following

(a) If \( A \subseteq C \) and \( B \subseteq C \), then \( A \cup B \subseteq C \).

(b) If \( C \subseteq A \) and \( C \subseteq B \), then \( C \subseteq A \cap B \).

**Solution:**

(a) *Proof:* If \( x \in A \cup B \), then either \( x \in A \) or \( x \in B \). If \( x \in A \) then \( x \in C \), since \( A \subseteq C \). Similarly, if \( x \in B \) then \( x \in C \) since \( B \subseteq C \). In either case, \( x \in C \). We have therefore shown that
\[
x \in A \cup B \Rightarrow x \in C;
\]
that is, \( A \cup B \subseteq C \).

(b) *Proof:* If \( x \in C \), then \( x \in A \) and \( x \in B \) since both \( c \subseteq A \) and \( C \subseteq B \) are assumed to be true. It then follows that \( x \in A \cap B \). We have thus shown that
\[
x \in C \Rightarrow x \in A \cap B;
\]
that is, \( C \subseteq A \cap B \).

2. Let \( \mathcal{C} \) be a sample space and \( \mathcal{B} \) be a \( \sigma \)-field of subsets of \( \mathcal{C} \). Prove that if \( \{E_1, E_2, E_3, \ldots\} \) is a sequence of events in \( \mathcal{B} \), then
\[
\bigcap_{k=1}^{\infty} E_k \in \mathcal{B}.
\]

**Hint:** Use De Morgan’s Laws.

**Proof:** Let \( E_1, E_2, E_3, \ldots \) be a sequence of events in \( \mathcal{B} \). Then, \( E_1^c, E_2^c, E_3^c, \ldots \) are also in \( \mathcal{B} \), and therefore
\[
\bigcup_{k=1}^{\infty} E_k^c \in \mathcal{B},
\]
and consequently,
\[
\left( \bigcup_{k=1}^{\infty} E_k^c \right)^c \in \mathcal{B}.
\]
It then follows by De Morgan’s laws that
\[ \bigcap_{k=1}^{\infty} (E_k^c)^c \in \mathcal{B}, \]
or
\[ \bigcap_{k=1}^{\infty} E_k \in \mathcal{B}. \]

3. Let \( \mathcal{C} \) be a sample space and \( \mathcal{B} \) be a \( \sigma \)-field of subsets of \( \mathcal{C} \). For fixed \( B \in \mathcal{B} \) define the collection of subsets

\[ \mathcal{B}_B = \{ D \subset \mathcal{C} \mid D = E \cap B \text{ for some } E \in \mathcal{B} \}. \]

Show that \( \mathcal{B}_B \) is a \( \sigma \)-field.

*Note:* In this case, the complement of \( D \in \mathcal{B}_B \) has to be understood as \( B \setminus D \); that is, the complement relative to \( B \). The \( \sigma \)-field \( \mathcal{B}_B \) is the \( \sigma \)-field \( \mathcal{B} \) restricted to \( B \), or *conditioned on* \( B \).

**Solution:** We verify that \( \mathcal{B}_B \) satisfies the three properties of a \( \sigma \)-field.

(i) Observe that \( \emptyset = \emptyset \cap B \), where \( \emptyset \in \mathcal{B} \). Thus, \( \emptyset \in \mathcal{B}_B \).

(ii) Let \( D \in \mathcal{B}_B \). Then,

\[ D = E \cap B \text{ for some } E \in \mathcal{B}. \]

Then, the complement of \( D \) relative to \( B \) is

\[
B \setminus D = B \setminus (E \cap B) \\
= B \cap (E \cap B)^c \\
= B \cap (E^c \cup B^c) \\
= (B \cap E^c) \cup (B \cap B^c) \\
= (B \cap E^c) \cup \emptyset \\
= B \cap E^c.
\]

Thus, \( B \setminus D = E^c \cap B \), where \( E^c \in \mathcal{B} \). It then follows that \( B \setminus D \in \mathcal{B}_B \).
(iii) Let $D_1, D_2, D_3, \ldots$ denote a sequence of events in $\mathcal{B}_B$. Then, there
exists a sequence $E_1, E_2, E_3, \ldots$ in $\mathcal{B}$ such that $D_k = E_k \cap B$ for all $k = 1, 2, 3, \ldots$
Then, by the distributive law,
\[
\bigcup_{k=1}^{\infty} D_k = \bigcup_{k=1}^{\infty} (E_k \cap B) = \left( \bigcup_{k=1}^{\infty} E_k \right) \cap B,
\]
where \( \bigcup_{k=1}^{\infty} E_k \in \mathcal{B} \).
It then follows that \( \bigcup_{k=1}^{\infty} D_k \in \mathcal{B}_B \).
\[\Box\]

4. Let $\mathcal{S}$ denote the collection of all bounded, open intervals $(a, b)$, where $a$ and $b$
are real numbers with $a < b$. Show that \( \mathcal{B}(\mathcal{S}) = \mathcal{B}_o \),
that is, the $\sigma$–field generated by bounded open intervals is the Borel $\sigma$–field.

Hints:

- We have already seen in the lecture that $\mathcal{B}_o$ contains all bounded open intervals.
- Observe also that the semi–infinite open interval $(b, \infty)$ can be expressed
  as the union of the sequence of bounded intervals $(b, k)$, for $k = 1, 2, 3, \ldots$

Proof: Let $\mathcal{S}$ denote the collection of all bounded, open intervals $(a, b)$, for
$a, b \in \mathbb{R}$ with $a < b$. We have proved in the lectures that
\( \mathcal{S} \subseteq \mathcal{B}_o \).
Since $\mathcal{B}(\mathcal{S})$ is the smallest $\sigma$–algebra which contains $\mathcal{S}$, it follows that
\( \mathcal{B}(\mathcal{S}) \subseteq \mathcal{B}_o \).
To show the reverse inclusion, first observe that, for any $b \in \mathbb{R}$,

$$(b, \infty) = \bigcup_{k=1}^{\infty} (b, k),$$

so that $(b, \infty) \in \mathcal{B}(S)$ for all $b \in \mathbb{R}$. It then follows that

$$(-\infty, b] = (b, \infty)^c \in \mathcal{B}(S) \quad \text{for all } b \in \mathbb{R}.$$ 

Since intervals of the form $(-\infty, b]$ generate the Borel $\sigma$–field $\mathcal{B}_o$, it follows that $\mathcal{B}_o \subseteq \mathcal{B}(S)$.

Combining this inclusion with the reverse inclusion that has been previously shown, we get that

$$\mathcal{B}(S) = \mathcal{B}_o.$$ 

\[\Box\]

5. Show that for every real number $a$, the singleton $\{a\}$ is in the Borel $\sigma$–field $\mathcal{B}_o$.

*Hint:* Express $\{a\}$ as an intersection of a sequence of open intervals.

*Proof:* We have seen in the previous problem that $\mathcal{B}_o$ is also generated by the bounded, open intervals of the form $(a, b)$. Thus, in view of Problem (1) in this set, we can prove that $\{a\}$ is in $\mathcal{B}_o$ by expressing it as an intersection of a sequence of such intervals.

Consider the intervals

$$E_k = \left(a - \frac{1}{k}, a + \frac{1}{k}\right), \quad \text{for } k = 1, 2, 3, \ldots$$

We claim that

$$\{a\} = \bigcap_{k=1}^{\infty} E_k.$$ 

To see why this is so, let $x \in \bigcap_{k=1}^{\infty} E_k$. Then, $x \in E_k$ for all $k$; that is,

$$a - \frac{1}{k} < x < a + \frac{1}{k} \quad \text{for all } k.$$ 

Since

$$\lim_{k \to \infty} \left(a - \frac{1}{k}\right) = \lim_{k \to \infty} \left(a + \frac{1}{k}\right) = a,$$

it follows from the Sandwich Theorem that $x = a$. This proves the claim. \[\Box\]