Assignment #1

Due on Wednesday, January 28, 2009

Read Chapter 2 on *Vector Algebra* in Bressoud (pp. 29–49).

Do the following problems

1. Let $\vec{v}_1 = \begin{pmatrix} -1 \\ 2 \\ -2 \end{pmatrix}$ and $\vec{v}_2 = \begin{pmatrix} 3 \\ -5 \\ 4 \end{pmatrix}$.

   (a) Give the parametric equations of the line through the point $P\colon (0,4,7)$ in the direction of the vector $\vec{v}_1$.

   (b) Give the equation of the plane through the point $P\colon (0,4,7)$ spanned by the vectors $\vec{v}_1$ and $\vec{v}_2$.

2. The following give parametric equations to two lines in $\mathbb{R}^3$:

$$\begin{cases} x = -1 + 4t \\ y = -7t \\ z = 2 - t \end{cases} \qquad \begin{cases} x = -1 + s \\ y = 2 - s \\ z = 2s \end{cases}$$

   Determine if the two lines ever meet. Justify your answer. If the lines do meet, give the equation of the plane that contains both lines.

3. The following give parametric equations to two lines in $\mathbb{R}^3$:

$$\begin{cases} x = 2 + 4t \\ y = -1 - 7t \\ z = 2 - t \end{cases} \qquad \begin{cases} x = s \\ y = 1 - s \\ z = -2 + 2s \end{cases}$$

   Determine if the two lines ever meet. Justify your answer. If the lines do meet, give the equation of the plane that contains both lines.

4. The vectors $\vec{v}_1 = \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}$, $\vec{v}_2 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$ and $\vec{v}_3 = \begin{pmatrix} 3 \\ 4 \\ 1 \end{pmatrix}$ in $\mathbb{R}^3$ can span a line, a plane or the entire three dimensional space $\mathbb{R}^3$. Give the equation of the geometric object which they span.

5. Exercises 10 and 11 on page 50 in the text.

6. Recall that the dot product, or inner product, of two vectors in $\mathbb{R}^n$ is symmetric, bi–linear and positive definite; that is, for vectors v, v_1, v_2 and w in $\mathbb{R}^n$,

   (i) $v \cdot w = w \cdot v$

   (ii) $(c_1 v_1 + c_2 v_2) \cdot w = c_1 v_1 \cdot w + c_2 v_2 \cdot w$, and

   (iii) $v \cdot v \geqslant 0$ for all $v \in \mathbb{R}^n$ and $v \cdot v = 0$ if and only if v is the zero vector.

   Use these properties of the the inner product in $\mathbb{R}^n$ to derive the following properties of the norm $\|\cdot\|$ in $\mathbb{R}^n$, where

   $$\|v\| = \sqrt{v \cdot v} \quad \text{for all vectors } \ v \in \mathbb{R}^n.$$

   (a) $\|v\| \geqslant 0$ for all $v \in \mathbb{R}^n$ and $\|v\| = 0$ if and only if $v = \vec{0}$.

   (b) For a scalar c, $\|cv\| = |c|\|v\|$.

7. Recall the Cauchy-Schwarz inequality: For any vectors v and w in $\mathbb{R}^n$,

   $$|v \cdot w| \leqslant \|v\|\|w\|.$$

   Use this inequality to derive the triangle inequality: For any vectors v and w in $\mathbb{R}^n$,

   $$\|v + w\| \leqslant \|v\| + \|w\|.$$

   (*Suggestion:* Start with the expression $\|v + w\|^2$ and use the properties of the inner product to simplify it.)

8. Given two non–zero vectors v and w in $\mathbb{R}^n$, the cosine of the angle, θ, between the vectors can be defined by

   $$\cos\theta = \frac{v \cdot w}{\|v\|\|w\|}.$$

   Use the Cauchy-Schwarz inequality to justify why this definition makes sense.

9. Two vectors v and w in $\mathbb{R}^n$ are said to be *orthogonal* or perpendicular, if and only if $v \cdot w = 0$.

   Show that if v and w are orthogonal, then

   $$\|v + w\|^2 = \|v\|^2 + \|w\|^2.$$

   Give a geometric interpretation of this result in two–dimensional Euclidean space.

10. A vector u in $\mathbb{R}^n$ is said to be a unit vector if and only if $\|u\| = 1$. Let u be a unit vector in $\mathbb{R}^n$ and v be any vector in $\mathbb{R}^n$.

    (a) Give the parametric equation of the line through origin in the direction of u.

    (b) Let $f(t) = \|v - tu\|^2$ for all $t \in \mathbb{R}^n$. Explain why this function gives the square of the distance from the point at v to a point on the line through the origin in the direction of u.

    (c) Show that $f(t)$ is minimized when $t = v \cdot u$.