Assignment #5

Due on Wednesday, March 4, 2009

Read Section 7.4 on *The Derivative,* pp. 187–197, in Bressoud.

Do the following problems

1. Let f denote a real valued function defined on some open interval around $a \in \mathbb{R}$. Consider a line of slope m and equation

$$L(x) = f(a) + m(x - a) \quad \text{for all} \quad x \in \mathbb{R}.$$

   Suppose that this line if the best approximation to the function f at a in the sense that
$$\lim_{x \to a} \frac{|E(x)|}{|x - a|} = 0,$$
   where $E(x) = f(x) - L(x)$ for all x in the interval in which f is defined.

   Prove that f is differentiable at a, and that $f'(a) = m$.

2. Recall that a function $F \colon U \to \mathbb{R}^m$, where U is an open subset for $\mathbb{R}^n$, is said to be differentiable at $x \in U$ if and only if there exists a unique linear transformation $T_x \colon \mathbb{R}^n \to \mathbb{R}^m$ such that

$$\lim_{\|y-x\| \to 0} \frac{\|F(y) - F(x) - T_x(y - x)\|}{\|y - x\|} = 0.$$

   Prove that if F is differentiable at x, then it is also continuous at x.

   Give an example that shows that the converse of this assertion is not true

3. Let $f \colon \mathbb{R}^2 \to \mathbb{R}$ be defined by $f(x, y) = \sqrt{|xy|}$ for all $(x, y) \in \mathbb{R}^2$. Show that f is not differentiable at $(0, 0)$.

4. Exercise 4 on page 197 in the text.

5. Exercise 6 on page 197 in the text.

6. Exercises 7(a) and 7(b) on pages 197 and 198 in the text.

7. Exercise 7(c) on page 198 in the text.

8. Exercise 8 on page 198 in the text.

9. Exercise 14 on pages 198 and 199 in the text.

10. A set $U \subseteq \mathbb{R}^n$ is said to be **path connected** iff for any pair of vectors x and y in U, there exists a differentiable path $\sigma\colon [0,1] \to \mathbb{R}^n$ such that $\sigma(0) = x$, $\sigma(1) = y$ and $\sigma(t) \in U$ for all $t \in [0,1]$; that is, any two elements in U can be connected by a differentiable path whose image is entirely contained in U.

    (a) Prove that the ball $B_R(O) = \{x \in \mathbb{R}^n \mid \|x\| < R\}$ is path connected.

    (b) Give and example in $\mathbb{R}^2$ of a set which is not path connected.