Assignment #6

Due on Wednesday, April 1, 2009

Read Section 7.4 on *The Derivative*, pp. 187–197, in Bressoud.

Read Section 7.6 on *The Chain Rule*, pp. 201–205, in Bressoud.

Do the following problems

1. Let $I$ be an open interval of real numbers, and suppose that $\sigma: I \to \mathbb{R}^n$ is a differentiable path satisfying $\sigma(t) \neq 0$ for all $t \in I$. Show that the function $g: I \to \mathbb{R}$ defined by $g(t) = \|\sigma(t)\|$ for all $t \in I$ is differentiable on $I$ and compute its derivative.

2. Recall that a set $U \subseteq \mathbb{R}^n$ is said to be path connected iff for any vectors $x$ and $y$ in $U$, there exists a differentiable path $\sigma: [0, 1] \to \mathbb{R}^n$ such that $\sigma(0) = x$, $\sigma(1) = y$ and $\sigma(t) \in U$ for all $t \in [0, 1]$; i.e., any two elements in $U$ can be connected by a differentiable path whose image is entirely contained in $U$.

Suppose that $U$ is an open, path connected subset of $\mathbb{R}^n$. Let $f: U \to \mathbb{R}$ be a differentiable scalar field such that $\nabla f(x)$ is the zero vector for all $x \in U$. Prove that $f$ must be constant.

3. Let $I$ be an open interval of real numbers and $U$ be an open subset of $\mathbb{R}^n$. Suppose that $\sigma: I \to \mathbb{R}^n$ is a differentiable path and that $f: U \to \mathbb{R}$ is a differentiable scalar field. Assume also that the image of $I$ under $\sigma$, $\sigma(I)$, is contained in $U$. Suppose also that the derivative of the path $\sigma$ satisfies

$$\sigma'(t) = -\nabla f(\sigma(t)) \quad \text{for all} \quad t \in I.$$  

Show that if the gradient of $f$ along the path $\sigma$ is never zero, then $f$ decreases along the path as $t$ increases.

*Suggestion:* Use the Chain Rule to compute the derivative of $f(\sigma(t))$.

4. Exercises 2 and 4 on page 207 in the text.

5. Exercise 6 on page 208 in the text.

6. Exercise 8 on page 208 in the text.
7. Let $D$ denote an open region in $\mathbb{R}^2$ and $f: D \to \mathbb{R}$ be a $C^2$ scalar field on $D$. The Jacobian of the gradient map $\nabla f: \mathbb{R}^2 \to \mathbb{R}^2$ is called the Hessian of the function $f$ and is denoted by $H_f$; that is $H_f(x,y) = J\nabla_f(x,y)$.

Compute the Hessian for the following scalar fields in $\mathbb{R}^2$.

(a) $f(x,y) = x^2 - y^2$ for all $(x,y) \in \mathbb{R}^2$.
(b) $f(x,y) = xy$ for all $(x,y) \in \mathbb{R}^2$.

8. Let $A$ denote a symmetric $n \times n$ matrix; recall that this means that $A^T = A$, where $A^T$ denotes the transpose of $A$. Define $f: \mathbb{R}^n \to \mathbb{R}$ by $f(x) = \frac{1}{2}(Ax) \cdot x$ for all $x \in \mathbb{R}^n$; that is, $f(x)$ is the dot–product of $Ax$ and $x$. In terms of matrix product,

$$f(x) = \frac{1}{2}(Ax)^T x \quad \text{for all} \quad x \in \mathbb{R}^n,$$

where $x$ is expressed as a column vector.

(a) Show that $f$ is differentiable and compute the gradient map $\nabla f$.

(b) Show that the gradient map $\nabla f$ is differentiable, and compute its derivative.

9. Let $U$ be an open subset of $\mathbb{R}^n$ and $I$ be an open interval. Suppose that $f: U \to \mathbb{R}$ is a differentiable scalar field and $\sigma: I \to \mathbb{R}^n$ be a differentiable path whose image lies in $U$. Suppose also that $\sigma'(t)$ is never the zero vector. Show that if $f$ has a local maximum or a local minimum at some point on the path, then $\nabla f$ is perpendicular to the path at that point.

Suggestion: Consider the real valued function of a single variable $g(t) = f(\sigma(t))$ for all $t \in I$.

10. Let $\sigma: [a,b] \to \mathbb{R}^n$ be a differentiable, one–to–one path. Suppose also that $\sigma'(t)$, is never the zero vector. Let $h: [c,d] \to [a,b]$ be a one–to–one and onto map such that $h'(t) \neq 0$ for all $t \in [c,d]$. Define

$$\gamma(t) = \sigma(h(t)) \quad \text{for all} \quad t \in [c,d].$$

$\gamma: [c,d] \to \mathbb{R}^n$ is a called a reparametrization of $\sigma$.

(a) Show that $\gamma$ is a differentiable, one–to–one path.

(b) Compute $\gamma'(t)$ and show that it is never the zero vector.

(c) Show that $\sigma$ and $\gamma$ have the same image in $\mathbb{R}^n$. 