Assignment #6

Due on Wednesday, April 1, 2009

Read Section 7.4 on *The Derivative,* pp. 187–197, in Bressoud.

Read Section 7.6 on *The Chain Rule,* pp. 201–205, in Bressoud.

Do the following problems

1. Let I be and open interval of real numbers, and suppose that $\sigma\colon I \to \mathbb{R}^n$ is a differentiable path satisfying $\sigma(t) \neq \mathbf{0}$ for all $t \in I$. Show that the function $g\colon I \to \mathbb{R}$ defined by $g(t) = \|\sigma(t)\|$ for all $t \in I$ is differentiable on I and compute its derivative.

2. Recall that a set $U \subseteq \mathbb{R}^n$ is said to be **path connected** iff for any vectors x and y in U, there exists a differentiable path $\sigma\colon [0,1] \to \mathbb{R}^n$ such that $\sigma(0) = x$, $\sigma(1) = y$ and $\sigma(t) \in U$ for all $t \in [0,1]$; i.e., any two elements in U can be connected by a differentiable path whose image is entirely contained in U.

   Suppose that U is an open, path connected subset of $\mathbb{R}^n$. Let $f\colon U \to \mathbb{R}$ be a differentiable scalar field such that $\nabla f(x)$ is the zero vector for all $x \in U$. Prove that f must be constant.

3. Let I be an open interval of real numbers and U ba an open subset of $\mathbb{R}^n$. Suppose that $\sigma\colon I \to \mathbb{R}^n$ is a differentiable path and that $f\colon U \to \mathbb{R}$ is a differentiable scalar field. Assume also that the image of I under σ, $\sigma(I)$, is contained in U. Suppose also that the derivative of the path σ satisfies

   $$\sigma'(t) = -\nabla f(\sigma(t)) \quad \text{for all} \quad t \in I.$$

   Show that if the gradient of f along the path σ is never zero, then f decreases along the path as t increases.

   Suggestion: Use the Chain Rule to compute the derivative of $f(\sigma(t))$.

4. Exercises 2 and 4 on page 207 in the text.

5. Exercise 6 on page 208 in the text.

6. Exercise 8 on page 208 in the text.

7. Let D denote an open region in $\mathbb{R}^2$ and $f\colon D \to \mathbb{R}$ be a C^2 scalar field on D. The Jacobian of the gradient map $\nabla f\colon \mathbb{R}^2 \to \mathbb{R}^2$ is called the *Hessian* of the function f and is denoted by H_f; that is $H_f(x,y) = J_{\nabla f}(x,y)$.

   Compute the Hessian for the following scalar fields in $\mathbb{R}^2$.

   (a) $f(x,y) = x^2 - y^2$ for all $(x,y) \in \mathbb{R}^2$.

   (b) $f(x,y) = xy$ for all $(x,y) \in \mathbb{R}^2$.

8. Let A denote a symmetric $n \times n$ matrix; recall that this means that $A^T = A$, where A^T denotes the transpose of A. Define $f\colon \mathbb{R}^n \to \mathbb{R}$ by $f(x) = \dfrac{1}{2}(Ax)\cdot x$ for all $x \in \mathbb{R}^n$; that is, $f(x)$ is the dot–product of Ax and x. In terms of matrix product,

$$f(x) = \frac{1}{2}(Ax)^T x \quad \text{for all} \quad x \in \mathbb{R}^n,$$

   where x is expressed as a column vector.

   (a) Show that f is differentiable and compute the gradient map ∇f.

   (b) Show that the gradient map ∇f is differentiable, and compute its derivative.

9. Let U be an open subset of $\mathbb{R}^n$ and I be an open interval. Suppose that $f\colon U \to \mathbb{R}$ is a differentiable scalar field and $\sigma\colon I \to \mathbb{R}^n$ be a differentiable path whose image lies in U. Suppose also that $\sigma'(t)$ is never the zero vector. Show that if f has a local maximum or a local minimum at some point on the path, then ∇f is perpendicular to the path at that point.

   Suggestion: Consider the real valued function of a single variable $g(t) = f(\sigma(t))$ for all $t \in I$.

10. Let $\sigma\colon [a,b] \to \mathbb{R}^n$ be a differentiable, one–to–one path. Suppose also that $\sigma'(t)$, is never the zero vector. Let $h\colon [c,d] \to [a,b]$ be a one–to–one and onto map such that $h'(t) \neq 0$ for all $t \in [c,d]$. Define

$$\gamma(t) = \sigma(h(t)) \quad \text{for all} \quad t \in [c,d].$$

   $\gamma\colon [c,d] \to \mathbb{R}^n$ is a called a *reparametrization* of σ

   (a) Show that γ is a differentiable, one–to–one path.

   (b) Compute $\gamma'(t)$ and show that it is never the zero vector.

   (c) Show that σ and γ have the same image in $\mathbb{R}^n$.