Assignment #8

Due on Wednesday, April 15, 2009

Read Section 3.1 on *The Calculus of Curves,* pp. 53–65, in Bressoud.

Read Section 5.2 on *Line Integrals,* pp. 113–119, in Bressoud.

Do the following problems

1. Consider a portion of a helix, C, parametrized by the path
$$\sigma(t) = (\cos t, t, \sin t) \quad \text{for} \quad 0 \leqslant t \leqslant \pi.$$
Let $F(x, y, z) = x\ \widehat{i} + y\ \widehat{j} + z\ \widehat{k}$, for all $(x, y, z) \in \mathbb{R}^3$, be a vector field in $\mathbb{R}^3$. Evaluate the line integral $\int_C F \cdot T$; that is, the integral of the tangential component of the field F along the curve C.

2. Evaluate
$$\int_C yz\ \mathrm{d}x + xz\ \mathrm{d}y + xy\ \mathrm{d}z$$
where C is the directed line segment from the point $(1, 1, 0)$ to the point $(3, 2, 1)$ in $\mathbb{R}^3$.

3. Exercises 1(a)(b)(c) on page 119 in the text.

4. Exercises 1(d)(e)(f) on page 119 in the text.

5. Let $f: U \to \mathbb{R}$ be a C^1 scalar field defined on an open subset U of $\mathbb{R}^n$. Define the vector field $F: U \to \mathbb{R}^n$ by $F(x) = \nabla f(x)$ for all $x \in U$. Suppose that C is a C^1 simple curve in U connecting the point x to the point y in U. Show that
$$\int_C F \cdot T = f(y) - f(x).$$
Conclude therefore that the line integral of F along a path from x to y in U is independent of the path connecting x to y. The field F is called a *gradient field.*

6. Exercise 4 on page 119 in the text.

7. Exercises 6(d)(e)(f) on pages 119 and 120 in the text.

8. Let $\sigma\colon [a,b] \to \mathbb{R}^n$ be a C^1 parametrization of a curve C in $\mathbb{R}^n$. Let $h\colon [c,d] \to [a,b]$ be a one–to–one and onto map such that $h'(t) > 0$ for all $t \in [c,d]$. Define

$$\gamma(t) = \sigma(h(t)) \quad \text{for all} \quad t \in [c,d].$$

$\gamma\colon [c,d] \to \mathbb{R}^n$ is a called a *reparametrization* of σ.

Let $F\colon U \to \mathbb{R}^n$ denote a continuous vector field defined on a region U of $\mathbb{R}^n$ which contains the curve C. Show that

$$\int_a^b F(\sigma(\tau)) \cdot \sigma'(\tau)\ \mathrm{d}\tau = \int_c^d F(\gamma(t)) \cdot \gamma'(t)\ \mathrm{d}t.$$

Thus, the line integral

$$\int_C F \cdot T\ \mathrm{d}s$$

is independent of reparametrization.

9. Let $\sigma\colon [0,1] \to \mathbb{R}^n$ be a C^1 parametrization of a curve C is $\mathbb{R}^n$. Give a C^1 reparametrization, $\gamma\colon [0,1] \to \mathbb{R}^n$, of σ in which the curve C is traversed in the opposite direction as that of σ. What is γ' in terms of σ'?

10. Recall that the flux of a 2–dimensional vector field,

$$F(x,y) = P(x,y)\,\widehat{i} + Q(x,y)\,\widehat{j},$$

across a simple, C^1, closed curve, C, is given by

$$\int_C P\ \mathrm{d}y - Q\ \mathrm{d}x.$$

Compute the flux of the following fields across the given curves

(a) $F(x,y) = x^2\,\widehat{i} + y^2\,\widehat{j}$ and C is the boundary of the square with vertices $(0,0)$, $(1,0)$, $(1,1)$ and $(0,1)$.

(b) $F(x,y) = x\,\widehat{i} + y\,\widehat{j}$ and C is the boundary of the unit circle