Assignment #8

Due on Wednesday, April 15, 2009

Read Section 3.1 on The Calculus of Curves, pp. 53–65, in Bressoud.

Read Section 5.2 on Line Integrals, pp. 113–119, in Bressoud.

Do the following problems

1. Consider a portion of a helix, $C$, parametrized by the path
   $$\sigma(t) = (\cos t, t, \sin t) \quad \text{for} \quad 0 \leq t \leq \pi.$$ 

   Let $F(x, y, z) = x \hat{i} + y \hat{j} + z \hat{k}$, for all $(x, y, z) \in \mathbb{R}^3$, be a vector field in $\mathbb{R}^3$. Evaluate the line integral $\int_C F \cdot T$; that is, the integral of the tangential component of the field $F$ along the curve $C$.

2. Evaluate
   $$\int_C yz \, dx + xz \, dy + xy \, dz$$

   where $C$ is the directed line segment from the point $(1, 1, 0)$ to the point $(3, 2, 1)$ in $\mathbb{R}^3$.

3. Exercises 1(a)(b)(c) on page 119 in the text.

4. Exercises 1(d)(e)(f) on page 119 in the text.

5. Let $f: U \to \mathbb{R}$ be a $C^1$ scalar field defined on an open subset $U$ of $\mathbb{R}^n$. Define the vector field $F: U \to \mathbb{R}^n$ by $F(x) = \nabla f(x)$ for all $x \in U$. Suppose that $C$ is a $C^1$ simple curve in $U$ connecting the point $x$ to the point $y$ in $U$. Show that

   $$\int_C F \cdot T = f(y) - f(x).$$

   Conclude therefore that the line integral of $F$ along a path from $x$ to $y$ in $U$ is independent of the path connecting $x$ to $y$. The field $F$ is called a gradient field.
6. Exercise 4 on page 119 in the text.

7. Exercises 6(d)(e)(f) on pages 119 and 120 in the text.

8. Let $\sigma : [a, b] \to \mathbb{R}^n$ be a $C^1$ parametrization of a curve $C$ in $\mathbb{R}^n$. Let $h : [c, d] \to [a, b]$ be a one-to-one and onto map such that $h'(t) > 0$ for all $t \in [c, d]$. Define

$$\gamma(t) = \sigma(h(t)) \text{ for all } t \in [c, d].$$

$\gamma : [c, d] \to \mathbb{R}^n$ is called a reparametrization of $\sigma$.

Let $F : U \to \mathbb{R}^n$ denote a continuous vector field defined on a region $U$ of $\mathbb{R}^n$ which contains the curve $C$. Show that

$$\int_a^b F(\sigma(\tau)) \cdot \sigma'(\tau) \, d\tau = \int_c^d F(\gamma(t)) \cdot \gamma'(t) \, dt.$$

Thus, the line integral

$$\int_C F \cdot T \, ds$$

is independent of reparametrization.

9. Let $\sigma : [0, 1] \to \mathbb{R}^n$ be a $C^1$ parametrization of a curve $C$ in $\mathbb{R}^n$. Give a $C^1$ reparametrization, $\gamma : [0, 1] \to \mathbb{R}^n$, of $\sigma$ in which the curve $C$ is traversed in the opposite direction as that of $\sigma$. What is $\gamma'$ in terms of $\sigma'$?

10. Recall that the flux of a 2–dimensional vector field,

$$F(x, y) = P(x, y) \hat{i} + Q(x, y) \hat{j},$$

across a simple, $C^1$, closed curve, $C$, is given by

$$\int_C P \, dy - Q \, dx.$$

Compute the flux of the following fields across the given curves

(a) $F(x, y) = x^2 \hat{i} + y^2 \hat{j}$ and $C$ is the boundary of the square with vertices $(0, 0)$, $(1, 0)$, $(1, 1)$ and $(0, 1)$.

(b) $F(x, y) = x \hat{i} + y \hat{j}$ and $C$ is the boundary of the unit circle.