Review Problems for Exam 1

1. Compute the (shortest) distance from the point \( P(4, 0, -7) \) in \( \mathbb{R}^3 \) to the plane given by
   \[
   4x - y - 3z = 12.
   \]

2. Compute the (shortest) distance from the point \( P(4, 0, -7) \) in \( \mathbb{R}^3 \) to the line given by the parametric equations
   \[
   \begin{cases}
   x &= -1 + 4t, \\
   y &= -7t, \\
   z &= 2 - t.
   \end{cases}
   \]

3. Compute the area of the triangle whose vertices in \( \mathbb{R}^3 \) are the points \((1, 1, 0), (2, 0, 1)\) and \((0, 3, 1)\)

4. Let \( v \) and \( w \) be two vectors in \( \mathbb{R}^3 \), and let \( \lambda \) be a scalar. Show that the area of the parallelogram determined by the vectors \( v \) and \( w + \lambda v \) is the same as that determined by \( v \) and \( w \).

5. Let \( \widehat{u} \) denote a unit vector in \( \mathbb{R}^n \) and \( P_{\widehat{u}}(v) \) denote the orthogonal projection of \( v \) along the direction of \( \widehat{u} \) for any vector \( v \in \mathbb{R}^n \). Use the Cauchy–Schwarz inequality to prove that the map
   \[
   v \mapsto P_{\widehat{u}}(v) \quad \text{for all} \quad v \in \mathbb{R}^n
   \]
   is a continuous map from \( \mathbb{R}^n \) to \( \mathbb{R}^n \).

6. Define the scalar field \( f: \mathbb{R}^n \rightarrow \mathbb{R} \) by \( f(v) = \frac{1}{2}\|v\|^2 \) for all \( v \in \mathbb{R}^n \). Show that \( f \) is differentiable on \( \mathbb{R}^n \) and compute the linear map \( Df(u): \mathbb{R}^n \rightarrow \mathbb{R} \) for all \( u \in \mathbb{R}^n \). What is the gradient of \( f \) at \( u \) for all \( x \in \mathbb{R}^n \)?

7. Let \( g: [0, \infty) \rightarrow \mathbb{R} \) be a differentiable, real–valued function of a single variable, and let \( f(x, y) = g(r) \) where \( r = \sqrt{x^2 + y^2} \).
   
   (a) Compute \( \frac{\partial r}{\partial x} \) in terms of \( x \) and \( r \), and \( \frac{\partial r}{\partial y} \) in terms of \( y \) and \( r \).
   
   (b) Compute \( \nabla f \) in terms of \( g'(r) \), \( r \) and the vector \( \mathbf{r} = \widehat{x}i + \widehat{y}j \).
8. Let \( f : U \to \mathbb{R} \) denote a scalar field defined on an open subset \( U \) of \( \mathbb{R}^n \), and let \( \hat{u} \) be a unit vector in \( \mathbb{R}^n \). If the limit
\[
\lim_{t \to 0} \frac{f(v + tu) - f(v)}{t}
\]
exists, we call it the direction derivative of \( f \) at \( v \) in the direction of the unit vector \( \hat{u} \). We denote it by \( D_{\hat{u}}f(v) \).

(a) Show that if \( f \) is differentiable at \( v \in U \), then, for any unit vector \( \hat{u} \) in \( \mathbb{R}^n \), the directional derivative of \( f \) in the direction of \( \hat{u} \) at \( v \) exists, and
\[
D_{\hat{u}}f(v) = \nabla f(v) \cdot \hat{u},
\]
where \( \nabla f(v) \) is the gradient of \( f \) at \( v \).

(b) Suppose that \( f : U \to \mathbb{R} \) is differentiable at \( v \in U \). Prove that if \( D_{\hat{u}}f(v) = 0 \) for every unit vector \( \hat{u} \) in \( \mathbb{R}^n \), then \( \nabla f(v) \) must be the zero vector.

(c) Suppose that \( f : U \to \mathbb{R} \) is differentiable at \( v \in U \). Use the Cauchy–Schwarz inequality to show that the largest value of \( D_{\hat{u}}f(v) \) is \( \| \nabla f(v) \| \) and it occurs when \( \hat{u} \) is in the direction of \( \nabla f(v) \).

9. The scalar field \( f : U \to \mathbb{R} \) is said to have a local minimum at \( x \in U \) if there exists \( r > 0 \) such that \( B_r(x) \subseteq U \) and
\[
f(x) \leq f(y) \quad \text{for every} \quad y \in B_r(x).
\]
Prove that if \( f \) is differentiable at \( x \in U \) and \( f \) has a local minimum at \( x \), then \( \nabla f(x) = 0 \), the zero vector in \( \mathbb{R}^n \).

10. Let \( I \) denote an open interval in \( \mathbb{R} \). Suppose that \( \sigma : I \to \mathbb{R}^n \) and \( \gamma : I \to \mathbb{R}^n \) are paths in \( \mathbb{R}^n \). Define a real valued function \( f : I \to \mathbb{R} \) of a single variable by
\[
f(t) = \sigma(t) \cdot \gamma(t) \quad \text{for all} \quad t \in I;
\]
that is, \( f(t) \) is the dot product of the two paths at \( t \).
Show that if \( \sigma \) and \( \gamma \) are both differentiable on \( I \), then so is \( f \), and
\[
f'(t) = \sigma'(t) \cdot \gamma(t) + \sigma(t) \cdot \gamma'(t) \quad \text{for all} \quad t \in I.
\]

11. Let \( \sigma : I \to \mathbb{R}^n \) denote a differentiable path in \( \mathbb{R}^n \). Show that if \( \| \sigma(t) \| \) is constant for all \( t \in I \), then \( \sigma'(t) \) is orthogonal to \( \sigma(t) \) for all \( t \in I \).
12. A particle is following a path in three–dimensional space given by

\[ \sigma(t) = (e^t, e^{-t}, 1 - t) \quad \text{for} \quad t \in \mathbb{R}. \]

At time \( t_0 = 1 \), the particle flies off on a tangent.

(a) Where will the particle be at time \( t_1 = 2 \)?

(b) Will the particle ever hit the \( xy \)–plane? If so, find the location on the \( xy \) plane where the particle hits.

13. Let \( U \) denote an open and convex subset of \( \mathbb{R}^n \). Suppose that \( f: U \to \mathbb{R} \) is differentiable at every \( x \in U \). Fix \( x \) and \( y \) in \( U \), and define \( g: [0, 1] \to \mathbb{R} \) by

\[ g(t) = f(x + t(y - x)) \quad \text{for} \quad 0 \leq t \leq 1. \]

(a) Explain why the function \( g \) is well defined.

(b) Show that \( g \) is differentiable on \((0, 1)\) and that

\[ g'(t) = \nabla f(x + t(y - x)) \cdot (y - x) \quad \text{for} \quad 0 < t < 1. \]

(\textit{Suggestion:} Consider \( \frac{g(t + h) - g(t)}{h} = \frac{f(x + t(y - x) + h(y - x)) - f(x + t(y - x))}{h} \)

and apply the definition of differentiability of \( f \) at the point \( x + t(y - x) \).)

(c) Use the Mean Value Theorem for derivatives to show that there exists a point \( z \) is the line segment connecting \( x \) to \( y \) such that

\[ f(y) - f(x) = D_{\hat{u}} f(z) \| y - x \|, \]

where \( \hat{u} \) is the unit vector in the direction of the vector \( y - x \); that is, \( \hat{u} = \frac{1}{\| y - x \|} (y - x) \).

(\textit{Hint:} Observe that \( g(1) - g(0) = f(y) - f(x) \).)

14. Prove that if \( U \) is an open and convex subset of \( \mathbb{R}^n \), and \( f: U \to \mathbb{R} \) is differentiable on \( U \) with \( \nabla f(v) = 0 \) for all \( v \in U \), then \( f \) must be a constant function.