Review Problems for Final Exam

1. In this problem, x and y denote vectors in $\mathbb{R}^n$.

   (a) Use the triangle inequality to derive the inequality
   $$|\ \|y\| - \|x\|\ | \leqslant \|y - x\| \quad \text{for all} \quad x, y \in \mathbb{R}^n.$$

   (b) Use the inequality derived in the previous part to show that the function $f\colon \mathbb{R}^n \to \mathbb{R}$ given by $f(x) = \|x\|$, for all $x \in \mathbb{R}^n$, is continuous.

   (c) Prove that the function $g\colon \mathbb{R}^n \to \mathbb{R}$ given by $g(x) = \sin(\|x\|)$, for all$x \in \mathbb{R}^n$, is continuous.

2. Define the scalar field $f\colon \mathbb{R}^n \to \mathbb{R}$ by $f(x) = \|x\|^2$ for all $x \in \mathbb{R}^n$.

   (a) Show that f is differentiable on $\mathbb{R}^n$ and compute the linear map
   $$Df(x)\colon \mathbb{R}^n \to \mathbb{R} \quad \text{for all} \quad x \in \mathbb{R}^n.$$
   What is the gradient of f at x for all $x \in \mathbb{R}^n$?

   (b) Let $\widehat{u}$ denote a unit vector in $\mathbb{R}^n$. For a fixed vector v in $\mathbb{R}^n$, define $g\colon \mathbb{R} \to \mathbb{R}$ by $g(t) = \|v - t\widehat{u}\|^2$, for all $t \in \mathbb{R}$. Show that g is differentiable and compute $g'(t)$ for all $t \in \mathbb{R}$.

   (c) Let $\widehat{u}$ be as in the previous part. For any $v \in \mathbb{R}^n$, give the point on the line spanned by $\widehat{u}$ which is the closest to v. Justify your answer.

3. For points $P_1(1,4,7)$, $P_2(7,1,4)$ and $P_3(4,7,1)$ in $\mathbb{R}^3$, define the oriented triangle $T = [P_1, P_2, P_3]$, and evaluate $\displaystyle\int_T \mathrm{d}x \wedge \mathrm{d}y$.

4. Let $\Phi\colon \mathbb{R}^2 \to \mathbb{R}^2$ denote the map from the uv–plane to the xy–plane given by
   $$\Phi\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 2u \\ v^2 \end{pmatrix} \quad \text{for all} \quad \begin{pmatrix} u \\ v \end{pmatrix} \in \mathbb{R}^2,$$
   and let T be the oriented triangle $[(0,0),(1,0),(1,1)]$ in the uv–plane.

   (a) Give the image, R, of the triangle T under the map Φ, and sketch it in the xy–plane.

   (b) Show that Φ is differentiable and give a formula for its derivative at every point $\begin{pmatrix} u \\ v \end{pmatrix}$ in $\mathbb{R}^2$.

5. Compute the arc length along the portion of the cycloid given by the parametric equations
$$x = t - \sin t \quad \text{and} \quad y = 1 - \cos t, \quad \text{for} \quad t \in \mathbb{R},$$
from the point $(0, 0)$ to the point $(2\pi, 0)$.

6. Evaluate the double integral $\int_R e^{-x^2} \mathrm{d}x\mathrm{d}y$, where R is the region in the xy–plane sketched in Figure 1.

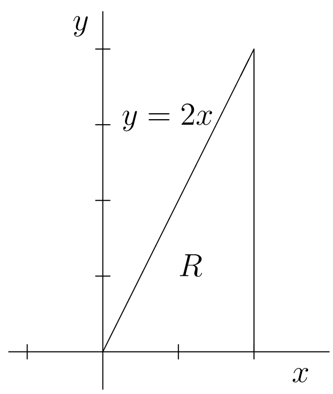

Figure 1: Sketch of Region R in Problem 6

7. Evaluate the line integral $\int_{\partial R} \omega$, where ω is the differential 1–form
$$\omega = (x^4 + y)\ \mathrm{d}x + (2x - y^4)\ \mathrm{d}y,$$
R is the rectangular region
$$R = \{(x, y) \in \mathbb{R}^2 \mid -1 \leqslant x \leqslant 3,\ -2 \leqslant y \leqslant 1\},$$
and ∂R is traversed in the counterclockwise sense.

8. Let $g \colon \mathbb{R}^3 \to \mathbb{R}$ be differentiable and define
$$S = g^{-1}(c) = \{(x, y, z) \in \mathbb{R}^3 \mid g(x, y, z) = c\}$$
for some constant c. Assume that $S \neq \emptyset$ and that $\nabla g(x, y, x) \neq \mathbf{0}$ for all $(x, y, z) \in S$. Let I be an open interval or real numbers and let $\sigma \colon I \to \mathbb{R}^3$ be a differentiable path satisfying $\sigma(t) \in S$ for all $t \in I$. Prove that $\nabla g(\sigma(t))$ is orthogonal to $\sigma'(t)$ for all $t \in I$.