Solutions to Review Problems for Final Exam

1. In this problem, x and y denote vectors in $\mathbb{R}^n$.

   (a) Use the triangle inequality to derive the inequality

   $$|\ \|y\| - \|x\|\ | \leqslant \|y - x\| \quad \text{for all} \quad x, y \in \mathbb{R}^n.$$

   Solution: Apply the triangle inequality to obtain

   $$\|x\| = \|(x - y) + y\| \leqslant \|x - y\| + \|y\|,$$

   from which we get that

   $$\|x\| - \|y\| \leqslant \|y - x\|, \tag{1}$$

   where we have used the fact that $\|y - x\| = \|x - y\|$. Similarly, from

   $$\|y\| = \|(y - x) + x\| \leqslant \|y - x\| + \|x\|,$$

   we get

   $$\|y\| - \|x\| \leqslant \|y - x\|. \tag{2}$$

   Combining (1) and (2) yields

   $$|\ \|y\| - \|x\|\ | \leqslant \|y - x\|. \tag{3}$$

   □

   (b) Use the inequality derived in the previous part to show that the function $f\colon \mathbb{R}^n \to \mathbb{R}$ given by $f(x) = \|x\|$, for all $x \in \mathbb{R}^n$, is continuous.

   Solution: Using the inequality in (3) we get

   $$0 \leqslant |f(y) - f(x)| \leqslant \|y - x\|.$$

   Thus, by the Squeeze Theorem, we get that

   $$\lim_{\|y-x\|\to 0} |f(y) - f(x)| = 0.$$

   which shows that f is continuous at x for every x in $\mathbb{R}^n$. □

   (c) Prove that the function $g\colon \mathbb{R}^n \to \mathbb{R}$ given by $g(x) = \sin(\|x\|)$, for all$x \in \mathbb{R}^n$, is continuous.

Solution: Note that $g = \sin\circ f$, where $f\colon \mathbb{R}^n \to \mathbb{R}$, given by $f(x) = \|x\|$ for all $x \in \mathbb{R}^n$, is continuous on $\mathbb{R}^n$ by the result in part (b). Thus, since $\sin\colon \mathbb{R} \to \mathbb{R}$ is continuous, it follows that g is continuous because it is the composition of two continuous functions. □

2. Define the scalar field $f\colon \mathbb{R}^n \to \mathbb{R}$ by $f(x) = \|x\|^2$ for all $x \in \mathbb{R}^n$.

   (a) Show that f is differentiable on $\mathbb{R}^n$ and compute the linear map

$$Df(x)\colon \mathbb{R}^n \to \mathbb{R} \quad \text{for all} \quad x \in \mathbb{R}^n.$$

   What is the gradient of f at x for all $x \in \mathbb{R}^n$?

Solution: For $w \in \mathbb{R}^n$, write

$$\begin{aligned} f(x+w) &= \|x+w\|^2 \\ &= (x+w\cdot x+w) \\ &= x\cdot x + x\cdot w + w\cdot x + w\cdot w \\ &= \|x\|^2 + 2x\cdot w + \|w\|^2. \end{aligned}$$

Consequently,

$$f(x+w) = f(x) + 2x\cdot w + E_x(w),$$

where $E_x(w) = \|w\|^2$ satisfies

$$\lim_{\|w\|\to 0} \frac{|E_x(w)|}{\|w\|} = 0.$$

Therefore, f is differentiable at x and the derivative map,

$$Df(x)\colon \mathbb{R}^n \to \mathbb{R},$$

of f at x is given by

$$Df(x)w = 2x\cdot w \quad \text{for all} \quad x \in \mathbb{R}^n.$$

We then have that the gradient of f at x is given by

$$\nabla f(x) = 2x \quad \text{for all} \quad x \in \mathbb{R}^n.$$

□

Alternate Solution: Write $x = (x_1, x_2, \ldots, x_n)$ so that

$$f(x) = x_1^2 + x_2^2 + \cdots + x_n^2 \quad \text{for all} \ \ x \in \mathbb{R}^n.$$

We then have that the partial derivatives of f at x exist and are given by

$$\frac{\partial f}{\partial x_i}(x) = 2x_i \quad \text{for} \ \ i = 1, 2, \ldots, n \ \ \text{and for all} \ \ x \in \mathbb{R}^n.$$

Thus, all the partial derivative of f at x are continuous and therefore f is a C^1 map. This implies that f is differentiable and its derivative is given by

$$Df(x)w = \nabla f(x) \cdot w \quad \text{for all} \ \ x \in \mathbb{R}^n,$$

where

$$\nabla f(x) = 2x \quad \text{for all} \ \ x \in \mathbb{R}^n.$$

□

(b) Let $\widehat{u}$ denote a unit vector in $\mathbb{R}^n$. For a fixed vector v in $\mathbb{R}^n$, define $g\colon \mathbb{R} \to \mathbb{R}$ by $g(t) = \|v - t\widehat{u}\|^2$, for all $t \in \mathbb{R}$. Show that g is differentiable and compute $g'(t)$ for all $t \in \mathbb{R}$.

Solution: Observe that $g = f \circ \sigma$ where f is given in part (a) and $\sigma\colon \mathbb{R} \to \mathbb{R}^n$ is the path given by

$$\sigma(t) = v - t\widehat{u} \quad \text{for all} \ \ t \in \mathbb{R}.$$

Note that σ is differentiable with derivative given by $\sigma'(t) = -\widehat{u}$ for all $t \in \mathbb{R}$. It then follows by the Chain Rule and part (a) that g is differentiable and its derivative is given by

$$g'(t) = Df(\sigma(t))\sigma'(t) = 2\sigma(t) \cdot \sigma'(t) \quad \text{for all} \ \ t \in \mathbb{R},$$

or

$$\begin{aligned} g'(t) &= 2(v - t\widehat{u}) \cdot (-\widehat{u}) \\ &= 2(-v \cdot \widehat{u} + t), \end{aligned}$$

since $\|\widehat{u}\| = 1$. □

(c) Let $\widehat{u}$ be as in the previous part. For any $v \in \mathbb{R}^n$, give the point on the line spanned by $\widehat{u}$ which is the closest to v. Justify your answer.

Solution: The point on the line spanned by $\widehat{u}$ which is the closest to v is a point determined by the vector $t_o\widehat{u}$, where $t_o \in \mathbb{R}$ at which the function $g(t) = \|v - t\widehat{u}\|^2$ is the smallest possible. Thus, we need to minimize the function g defined in part (b). Since this function is differentiable, we may first locate its critical points by solving

$$g'(t) = 0.$$

This yields $t_o = v \cdot \widehat{u}$. Note that sice $g''(t) = 2 > 0$, we get that $g(t_o)$ is a global minimum for g. Thus, the point on the line spanned by $\widehat{u}$ which is the closest to v is the point determined by the vector $(v \cdot \widehat{u})\widehat{u}$. □

3. For points $P_1(1, 4, 7)$, $P_2(7, 1, 4)$ and $P_3(4, 7, 1)$ in $\mathbb{R}^3$, define the oriented triangle $T = [P_1, P_2, P_3]$, and evaluate $\displaystyle\int_T \mathrm{d}x \wedge \mathrm{d}y$.

Solution: Define the vectors

$$v = \overrightarrow{P_1P_2} = \begin{pmatrix} 6 \\ -3 \\ -3 \end{pmatrix} \quad \text{and} \quad w = \overrightarrow{P_1P_3} = \begin{pmatrix} 3 \\ 3 \\ -6 \end{pmatrix}.$$

Then,

$$\int_T \mathrm{d}x \wedge \mathrm{d}y = \frac{1}{2}(v \times w) \cdot \widehat{k},$$

where

$$\begin{aligned} v \times w &= \begin{vmatrix} \widehat{i} & \widehat{j} & \widehat{k} \\ 6 & -3 & -3 \\ 3 & 3 & -6 \end{vmatrix} \\ &= \begin{vmatrix} -3 & -3 \\ 3 & -6 \end{vmatrix} \widehat{i} - \begin{vmatrix} 6 & -3 \\ 3 & -6 \end{vmatrix} \widehat{j} + \begin{vmatrix} 6 & -3 \\ 3 & 3 \end{vmatrix} \widehat{k} \\ &= 27\widehat{i} + 27\widehat{j} + 27\widehat{k}. \end{aligned}$$

Consequently,

$$\int_T \mathrm{d}x \wedge \mathrm{d}y = \frac{27}{2}.$$

□

4. Let $\Phi\colon \mathbb{R}^2 \to \mathbb{R}^2$ denote the map from the uv–plane to the xy–plane given by

$$\Phi\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 2u \\ v^2 \end{pmatrix} \quad \text{for all} \quad \begin{pmatrix} u \\ v \end{pmatrix} \in \mathbb{R}^2,$$

and let T be the oriented triangle $[(0,0),(1,0),(1,1)]$ in the uv–plane.

(a) Give the image, R, of the triangle T under the map Φ, and sketch it in the xy–plane.

Solution: The image of R under Φ is the set

$$\begin{aligned} \Phi(R) &= \{(x,y) \in \mathbb{R}^2 \mid x = 2u, y = v^2, \text{ for some } \ (u,v) \in \mathbb{R}\} \\ &= \{(x,y) \in \mathbb{R}^2 \mid 0 \leqslant x \leqslant 2,\ 0 \leqslant y \leqslant x^2/4\mathbb{R}\}. \end{aligned}$$

A sketch of $\Phi(R)$ is shown in Figure 1.

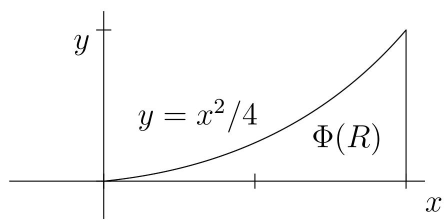

Figure 1: Sketch of Region $\Phi(R)$

□

(b) Show that Φ is differentiable and give a formula for its derivative at every point $\begin{pmatrix} u \\ v \end{pmatrix}$ in $\mathbb{R}^2$.

Solution: Write

$$\Phi\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} f(u,v) \\ g(u,v) \end{pmatrix} \quad \text{for all} \quad \begin{pmatrix} u \\ v \end{pmatrix} \in \mathbb{R}^2,$$

where $f(u,v) = 2u$ and $g(u,v) = v^2$ for all $\begin{pmatrix} u \\ v \end{pmatrix} \in \mathbb{R}^2$. Observe that the partial derivatives of f and g exist and are given by

$$\frac{\partial f}{\partial u}(u,v) = 2, \qquad \frac{\partial f}{\partial v}(u,v) = 0$$

$$\frac{\partial g}{\partial u}(u,v) = 0, \qquad \frac{\partial g}{\partial v}(u,v) = 2v.$$

Note that the partial derivatives of f and g are continuous. Therefore, Φ is a C^1 map. Hence, Φ is differentiable on $\mathbb{R}^2$ and its derivative map at (u, v), for any $(u, v) \in \mathbb{R}^2$ is given by multiplication by the Jacobian matrix

$$D\Phi(u, v) = \begin{pmatrix} 2 & 0 \\ 0 & 2v \end{pmatrix};$$

that is,

$$D\Phi(u, v)\begin{pmatrix} h \\ k \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 2v \end{pmatrix}\begin{pmatrix} h \\ k \end{pmatrix} = \begin{pmatrix} 2h \\ 2vk \end{pmatrix}$$

for all $\begin{pmatrix} h \\ k \end{pmatrix} \in \mathbb{R}^2$. □

5. Compute the arc length along the portion of the cycloid given by the parametric equations

$$x = t - \sin t \quad \text{and} \quad y = 1 - \cos t, \quad \text{for} \quad t \in \mathbb{R},$$

from the point $(0, 0)$ to the point $(2\pi, 0)$.

Solution: Put $\sigma(t) = (t - \sin t, 1 - \cos t)$ for $t \in \mathbb{R}$. Then,

$$\sigma'(t) = (1 - \cos t, \sin t) \quad \text{for all} \quad t \in \mathbb{R},$$

and therefore

$$\|\sigma'(t)\|^2 = (1 - \cos t)^2 + \sin^2 t \quad \text{for al } t \in \mathbb{R},$$

which may be simplified to

$$\begin{aligned} \|\sigma'(t)\|^2 &= 1 - 2\cos t + \cos^2 t + \sin^2 t \\ &= 2 - 2\cos t \\ &= 2(1 - \cos t) \\ &= 4\sin^2\left(\frac{t}{2}\right). \end{aligned}$$

Taking square roots on both sides we get that

$$\|\sigma'(t)\| = 2\left|\sin\left(\frac{t}{2}\right)\right| \quad \text{for all} \quad t \in \mathbb{R}.$$

Next, since $0 \leqslant \dfrac{t}{2} \leqslant \pi$ for $0 \leqslant t \leqslant 2\pi$, it follows that the arc length along the portion of the cycloid parametrized by $\sigma(t)$ for $0 \leqslant t \leqslant 2\pi$ is

$$\begin{aligned} \int_0^{2\pi} \|\sigma'(t)\| \, \mathrm{d}t &= \int_0^{2\pi} 2\sin\left(\frac{t}{2}\right) \, \mathrm{d}t \\ &= \left[-4\cos\left(\frac{t}{2}\right)\right]_0^{2\pi} \\ &= 8. \end{aligned}$$

□

6. Evaluate the double integral $\int_R e^{-x^2} \mathrm{d}x\mathrm{d}y$, where R is the region in the xy–plane sketched in Figure 2.

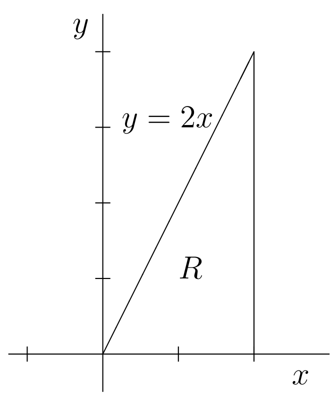

Figure 2: Sketch of Region R in Problem 6

***Solution**:* Compute

$$\begin{aligned}\int_R e^{-x^2}\ \mathrm{d}x\mathrm{d}y &= \int_0^2 \int_0^{2x} e^{-x^2}\ \mathrm{d}y\mathrm{d}x \\ &= \int_0^2 2xe^{-x^2}\ \mathrm{d}x \\ &= \left[-e^{-x^2}\right]_0^2 \\ &= 1 - e^{-4}.\end{aligned}$$

□

7. Evaluate the line integral $\int_{\partial R} \omega$, where ω is the differential 1–form

$$\omega = (x^4 + y)\ \mathrm{d}x + (2x - y^4)\ \mathrm{d}y,$$

R is the rectangular region

$$R = \{(x, y) \in \mathbb{R}^2 \mid -1 \leqslant x \leqslant 3,\ -2 \leqslant y \leqslant 1\},$$

and ∂R is traversed in the counterclockwise sense.

***Solution**:* Use the Fundamental Theorem of Calculus:

$$\int_{\partial R} \omega = \int_R \mathrm{d}\omega,$$

where

$$\begin{aligned}\mathrm{d}\omega &= \mathrm{d}(x^4 + y) \wedge \mathrm{d}x + \mathrm{d}(2x - y^4) \wedge \mathrm{d}y \\ &= (4x^3\mathrm{d}x + \mathrm{d}y) \wedge \mathrm{d}x + (2\mathrm{d}x - 4y^3\mathrm{d}y) \wedge \mathrm{d}y \\ &= \mathrm{d}y \wedge \mathrm{d}x + 2\mathrm{d}x \wedge \mathrm{d}y \\ &= \mathrm{d}x \wedge \mathrm{d}y,\end{aligned}$$

since $\mathrm{d}y \wedge \mathrm{d}x = -\mathrm{d}x \wedge \mathrm{d}y$. Consequently,

$$\int_{\partial R} \omega = \int_R \mathrm{d}x \wedge \mathrm{d}y = \mathrm{area}(R) = 12,$$

since R is a rectangle of dimensions 4 and 3 units. □

8. Let $g\colon \mathbb{R}^3 \to \mathbb{R}$ be differentiable and define

$$S = g^{-1}(c) = \{(x, y, z) \in \mathbb{R}^3 \mid g(x, y, z) = c\}$$

for some constant c. Assume that $S \neq \emptyset$ and that $\nabla g(x, y, x) \neq \mathbf{0}$ for all $(x, y, z) \in S$. Let I be an open interval or real numbers and let $\sigma\colon I \to \mathbb{R}^3$ be a differentiable path satisfying $\sigma(t) \in S$ for all $t \in I$. Prove that $\nabla g(\sigma(t))$ is orthogonal to $\sigma'(t)$ for all $t \in I$.

***Solution**:* Since $\sigma(t) \in S$ for all $t \in I$, it follows that

$$g(\sigma(t)) = c \quad \text{for all } t \in I.$$

Thus, differentiating with respect to t on both sides and applying the Chain Rule, we obtain that

$$\nabla g(\sigma(t)) \cdot \sigma'(t) = 0, \quad \text{for all } t \in I,$$

which shows that $\nabla g(\sigma(t))$ is orthogonal to $\sigma'(t)$ for all $t \in I$. □