Assignment #11

Due on Monday, March 2, 2009

Read Section 3.6 on *Coordinates* in Messer (pp. 123–127).

Background and Definitions

- *(Ordered Basis).* Let W be a subspace of $\mathbb{R}^n$ of dimension k and let B denote a basis for W. If the elements in B are listed in a specified order: $B = \{w_1, w_2, \ldots, w_k\}$, then B is called an **ordered basis**. In this sense, the basis $B_1 = \{w_2, w_1, \ldots, w_k\}$ is different from B even though, as sets, B and B_1 are the same; that is, the contain the same elements.

- *(Coordinates Relative to a Basis).* Let W be a subspace of $\mathbb{R}^n$ and
$$B = \{w_1, w_2, \ldots, w_k\}$$
be an ordered basis for W. Given any vector, v, in W, the **coordinates of** v **relative to the basis** B, are the unique set of scalars $c_1, c_2, \ldots, c_k$ such that
$$v = c_1 w_1 + c_2 w_2 + \cdots + c_k w_k.$$
We denote the coordinates of v relative to the basis B by the symbol $[v]_B$ and write $[v]_B = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_k \end{pmatrix}$. The vector $[v]_B$ in $\mathbb{R}^k$ is also called the **coordinates vector for** v **with respect to the basis** B.

Do the following problems

1. Let $W = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 \,\middle|\, 3x - 2y + z = 0 \right\}$.

   (a) Show that the set $B = \left\{ \begin{pmatrix} 1 \\ 0 \\ -3 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} \right\}$ is a basis for W.

   (b) Let $v = \begin{pmatrix} 2 \\ 3 \\ 0 \end{pmatrix}$. Show that $v \in W$ and compute $[v]_B$.

2. Suppose that B is an ordered basis for $\mathbb{R}^2$ satisfying

$$\left[\begin{pmatrix} 3 \\ 2 \end{pmatrix}\right]_B = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \text{and} \quad \left[\begin{pmatrix} -1 \\ 4 \end{pmatrix}\right]_B = \begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

   Determine the two vectors in the basis B.

3. Find a condition on the scalars a, b, c and d so that the columns of the matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

   are linearly independent in $\mathbb{R}^2$.

   Suggestion: Consider the cases $a = 0$ and $a \neq 0$ separately.

4. Let the matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ satisfy the condition you discovered in Problem 3. Prove that the columns of A span $\mathbb{R}^2$.

5. Let the matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ satisfy the condition you discovered in Problem 3 and denote the columns of A by C_1 and C_2, respectively; that is,

$$C_1 = \begin{pmatrix} a \\ c \end{pmatrix} \quad \text{and} \quad C_2 = \begin{pmatrix} b \\ d \end{pmatrix},$$

   Find the coordinates of any vector $v = \begin{pmatrix} x \\ y \end{pmatrix}$ in $\mathbb{R}^2$ with respect to the ordered basis $B = \{C_1, C_2\}$.