Assignment #11
Due on Monday, March 2, 2009

Read Section 3.6 on Coordinates in Messer (pp. 123–127).

Background and Definitions

• (Ordered Basis). Let $W$ be a subspace of $\mathbb{R}^n$ of dimension $k$ and let $B$ denote a basis for $W$. If the elements in $B$ are listed in a specified order: $B = \{w_1, w_2, \ldots, w_k\}$, then $B$ is called an ordered basis. In this sense, the basis $B_1 = \{w_2, w_1, \ldots, w_k\}$ is different from $B$ even though, as sets, $B$ and $B_1$ are the same; that is, the contain the same elements.

• (Coordinates Relative to a Basis). Let $W$ be a subspace of $\mathbb{R}^n$ and

$$B = \{w_1, w_2, \ldots, w_k\}$$

be an ordered basis for $W$. Given any vector, $v$, in $W$, the coordinates of $v$ relative to the basis $B$, are the unique set of scalars $c_1, c_2, \ldots, c_k$ such that

$$v = c_1w_1 + c_2w_2 + \cdots + c_kw_k.$$ 

We denote the coordinates of $v$ relative to the basis $B$ by the symbol $[v]_B$ and write $[v]_B = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_k \end{pmatrix}$. The vector $[v]_B$ in $\mathbb{R}^k$ is also called the coordinates vector for $v$ with respect to the basis $B$.

Do the following problems

1. Let $W = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 \mid 3x - 2y + z = 0 \right\}$.

   (a) Show that the set $B = \left\{ \begin{pmatrix} 1 \\ 0 \\ -3 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} \right\}$ is a basis for $W$.

   (b) Let $v = \begin{pmatrix} 2 \\ 3 \\ 0 \end{pmatrix}$. Show that $v \in W$ and compute $[v]_B$. 
2. Suppose that $B$ is an ordered basis for $\mathbb{R}^2$ satisfying
\[
\begin{pmatrix}
3 \\
2
\end{pmatrix}
_B = \begin{pmatrix}
1 \\
1
\end{pmatrix}
\quad \text{and} \quad
\begin{pmatrix}
-1 \\
4
\end{pmatrix}
_B = \begin{pmatrix}
2 \\
1
\end{pmatrix}.
\]
Determine the two vectors in the basis $B$.

3. Find a condition on the scalars $a$, $b$, $c$ and $d$ so that the columns of the matrix
\[
\begin{pmatrix}
a & b \\
c & d
\end{pmatrix}
\]
are linearly independent in $\mathbb{R}^2$.

*Suggestion:* Consider the cases $a = 0$ and $a \neq 0$ separately.

4. Let the matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ satisfy the condition you discovered in Problem 3. Prove that the columns of $A$ span $\mathbb{R}^2$.

5. Let the matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ satisfy the condition you discovered in Problem 3 and denote the columns of $A$ by $C_1$ and $C_2$, respectively; that is,
\[
C_1 = \begin{pmatrix} a \\ c \end{pmatrix} \quad \text{and} \quad
C_2 = \begin{pmatrix} b \\ d \end{pmatrix}.
\]
Find the coordinates of any vector $v = \begin{pmatrix} x \\ y \end{pmatrix}$ in $\mathbb{R}^2$ with respect to the ordered basis $B = \{C_1, C_2\}$. 