Assignment #15
Due on Monday, March 30, 2009

Read Section 1.6 on Matrices in Messer (pp. 29–31).

Read Section 5.1 on Matrix Algebra in Messer (pp. 176–182).

Do the following problems

1. Let $A$ be an $m \times n$ matrix, and $\{e_1, e_2, \ldots, e_n\}$ denote the standard basis in $\mathbb{R}^n$.
   (a) Prove that $Ae_j$ is the $j$th column of the matrix $A$.
   (b) Use your result from part (a) to prove that $AI = A$, where $I$ denotes the $n \times n$ identity matrix.

2. Recall that the null space of a matrix $A \in \mathbb{M}(m, n)$, denoted by $N_A$, is the space of solutions to the equation $Ax = 0$; that is, $N_A = \{v \in \mathbb{R}^n \mid Av = 0\}$. Prove that $v \in N_A$ if and only if $v$ is orthogonal to the rows of $A$.

3. Recall that the transpose of an $m \times n$ matrix, $A = [a_{ij}]$, is the $n \times m$ matrix $A^T$ given by $A^T = [a_{ji}]$, for $1 \leq i \leq m$ and $1 \leq j \leq n$.
   Let $A \in \mathbb{M}(m, n)$ and $B \in \mathbb{M}(n, k)$. Prove that $(AB)^T = B^T A^T$.

4. Consider any diagonal matrix $A = \begin{pmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{pmatrix} \in \mathbb{M}(3, 3)$.

   Prove that there exist constants $c_o$, $c_1$, $c_2$ and $c_3$ such that
   
   $$c_o I + c_1 A + c_2 A^2 + c_3 A^3 = O,$$

   where $I$ is the identity matrix in $\mathbb{M}(3, 3)$ and $O$ denotes the $3 \times 3$ zero–matrix.

   In other words, there exists a polynomial, $p(x) = c_o + c_1 x + c_2 x^2 + c_3 x^3$, of degree 3, such that $p(A) = O$.

5. Let $A = \begin{pmatrix} 1 & 2 & 1 \\ 0 & -2 & 3 \\ 4 & 1 & 2 \end{pmatrix}$.
   (a) Compute $A^2$ and $A^3$.
   (b) Verify that $A^3 - A^2 - 11A - 25I = O$, where $I$ is the identity matrix in $\mathbb{M}(3, 3)$ and $O$ denotes the $3 \times 3$ zero–matrix.
   (c) Use the result of part (b) above to find a matrix $B \in \mathbb{M}(3, 3)$ such that $AB = I$. 