Solutions to Assignment #15

1. Let $A$ be an $m \times n$ matrix, and $\{e_1, e_2, \ldots, e_n\}$ denote the standard basis in $\mathbb{R}^n$.

(a) Prove that $Ae_j$ is the $j^{\text{th}}$ column of the matrix $A$.

\textbf{Solution:} Write $A = \begin{pmatrix} R_1 \\ R_2 \\ \vdots \\ R_m \end{pmatrix}$, where $R_1, R_2, \ldots, R_m$ are the rows of $A$, and $e_j = \begin{pmatrix} \delta_1 \\ \delta_2 \\ \vdots \\ \delta_n \end{pmatrix}$, where $\delta_k = 1$ if $k = j$, but $\delta_k = 0$ if $k \neq j$. Then,

$$Ae_j = \begin{pmatrix} R_1e_j \\ R_2e_j \\ \vdots \\ R_me_j \end{pmatrix},$$

where, for each $i = 1, 2, \ldots, m$,

$$R_ie_j = \sum_{k=1}^{n} a_{ik}\delta_k = a_{ij}.$$

Thus,

$$Ae_j = \begin{pmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{mj} \end{pmatrix},$$

which is the $j^{\text{th}}$ column of the matrix $A$. \hfill \Box

(b) Use your result from part (a) to prove that $AI = A$, where $I$ denotes the $n \times n$ identity matrix.

\textbf{Solution:} Observe that the identity matrix in $\mathbb{M}(n, n)$ can be written as

$$I = \begin{bmatrix} e_1 & e_2 & \cdots & e_n \end{bmatrix}.$$

Then,

$$AI = [Ae_1 \ Ae_2 \ \cdots \ Ae_n] = A,$$

since $Ae_j$ is the $j^{\text{th}}$ column of $A$ for each $j = 1, 2, \ldots, n$. \hfill \Box
2. Recall that the null space of a matrix $A \in \mathbb{M}(m,n)$, denoted by $N_A$, is the space of solutions to the equation $Ax = 0$; that is, $N_A = \{ v \in \mathbb{R}^n \mid Av = 0 \}$. Prove that $v \in N_A$ if and only if $v$ is orthogonal to the rows of $A$.

**Solution:** Write $A = \begin{pmatrix} R_1 \\ R_2 \\ \vdots \\ R_m \end{pmatrix}$, where $R_1, R_2, \ldots, R_m$ are the rows of $A$. Observe that for any vector $v \in \mathbb{R}^n$,

$$Av = \begin{pmatrix} R_1 v \\ R_2 v \\ \vdots \\ R_m v \end{pmatrix},$$

where, for each $i = 1, 2, \ldots, m$,

$$R_i v = \langle R_i^T, v \rangle;$$

that is, $R_i v$ is the Euclidean inner product of the vectors $R_i^T$ and $v$.

It then follows that $v \in N_A$ if and only if

$$\langle R_i^T, v \rangle \quad \text{for all } i = 1, 2, \ldots, m;$$

that is, $v$ is orthogonal to the rows of $A$. \qed

3. Recall that the transpose of an $m \times n$ matrix, $A = [a_{ij}]$, is the $n \times m$ matrix $A^T$ given by $A^T = [a_{ji}]$, for $1 \leq i \leq m$ and $1 \leq j \leq n$.

Let $A \in \mathbb{M}(m,n)$ and $B \in \mathbb{M}(n,k)$. Prove that $(AB)^T = B^T A^T$.

**Proof:** Write $A = [a_{ij}] \in \mathbb{M}(m,n)$ and $B = [b_{j\ell}] \in \mathbb{M}(n,k)$, where $1 \leq i \leq m$, $1 \leq j \leq n$ and $1 \leq \ell \leq k$. Put $A^T = [a'_{ji}]$ and $B^T = [b'_{\ell j}]$, where $a'_{ji} = a_{ij}$ and $b'_{\ell j} = b_{\ell j}$.

Next, compute $AB = [d_{i\ell}]$, where $d_{i\ell} = \sum_{j=1}^{n} a_{ij} b_{j\ell}$, for $1 \leq i \leq m$ and $1 \leq \ell \leq k$.

Consequently, $(AB)^T = [d'_{\ell i}]$, where $d'_{\ell i} = d_{i\ell}$. Note that

$$d'_{\ell i} = \sum_{j=1}^{n} a_{ij} b_{j\ell} = \sum_{j=1}^{n} a'_{ji} b'_{\ell j} = \sum_{j=1}^{n} b'_{\ell j} a'_{ji},$$
which shows that $d'_{\ell i}$, for $1 \leq \ell k$ and $1 \leq i \leq m$, are the entries in the matrix product $B^T A^T$; that is,

$$(AB)^T = B^T A^T,$$

which was to be shown.

4. Consider any diagonal matrix $A = \begin{pmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{pmatrix} \in M(3, 3)$.

Prove that there exist constants $c_0$, $c_1$, $c_2$ and $c_3$ such that

$$c_0 I + c_1 A + c_2 A^2 + c_3 A^3 = O,$$

where $I$ is the identity matrix in $M(3, 3)$ and $O$ denotes the $3 \times 3$ zero–matrix. In other words, there exists a polynomial, $p(x) = c_0 + c_1 x + c_2 x^2 + c_3 x^3$, of degree 3, such that $p(A) = O$.

**Proof:** Let $\mathcal{W}$ denote the set of all diagonal $3 \times 3$ matrices. Then, $\mathcal{W}$ is a subspace of $M(3, 3)$; it fact,

$$\mathcal{W} = \text{span} \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right\},$$

and consequently $\dim(\mathcal{W}) = 3$.

Observe also that the matrices $I$, $A$, $A^2$ and $A^3$ are in $\mathcal{W}$. Hence, since $\mathcal{W}$ has dimension 3, it follows that the set

$$\{I, A, A^2, A^3\}$$

is linearly independent. Therefore, there exist constants $c_0$, $c_1$, $c_2$ and $c_3$ such that

$$c_0 I + c_1 A + c_2 A^2 + c_3 A^3 = O,$$

which was to be shown.

5. Let $A = \begin{pmatrix} 1 & 2 & 1 \\ 0 & -2 & 3 \\ 4 & 1 & 2 \end{pmatrix}$.

(a) Compute $A^2$ and $A^3$. 

**Answer:** Compute

\[
A^2 = \begin{pmatrix}
5 & -1 & 9 \\
12 & 7 & 0 \\
12 & 8 & 11
\end{pmatrix},
\]

and

\[
A^3 = \begin{pmatrix}
41 & 21 & 20 \\
12 & 10 & 33 \\
56 & 19 & 58
\end{pmatrix},
\]

\(\square\)

(b) Verify that \(A^3 - A^2 - 11A - 25I = O\), where \(I\) is the identity matrix in \(\mathbb{M}(3,3)\) and \(O\) denotes the \(3 \times 3\) zero–matrix.

**Solution:** Compute \(A^3 - A^2 - 11A - 25I\) to get the \(3 \times 3\) zero–matrix.

\(\square\)

(c) Use the result of part (b) above to find a matrix \(B \in \mathbb{M}(3,3)\) such that \(AB = I\).

**Solution:** Start with the equation

\[A^3 - A^2 - 11A - 25I = O,\]

add \(25I\) on both sides and write \(A = AI\) to get

\[A^3 - A^2 - 11AI = 25I.\]

Applying the distributive property on the left–hand side to factor out \(A\) we obtain

\[A(A^2 - A - 11I) = 25I.\]

Thus, multiplying on both sides by \(1/25\),

\[A \left[ \frac{1}{25} (A^2 - A - 11I) \right] = I.\]

Thus, we see that

\[B = \frac{1}{25} (A^2 - A - 11I),\]

where

\[A^2 - A2 - 11I = \begin{pmatrix}
-7 & -3 & 8 \\
12 & -2 & -3 \\
8 & 7 & -2
\end{pmatrix}.
\]
It then follows that

\[
B = \frac{1}{25} \begin{pmatrix}
-7 & -3 & 8 \\
12 & -2 & -3 \\
8 & 7 & -2
\end{pmatrix},
\]

or

\[
B = \begin{pmatrix}
-7/25 & -3/25 & 8/25 \\
8/25 & 7/25 & -2/25
\end{pmatrix},
\]