Solutions to Assignment #16

1. Let $A$ denote an $m \times n$ matrix and let $\{e_1, e_2, \ldots, e_n\}$ denote the standard basis in $\mathbb{R}^n$.

(a) Prove that if $A$ has a left–inverse, $B$, then the set $\{Ae_1, Ae_2, \ldots, Ae_n\}$ is a linearly independent subset of $\mathbb{R}^m$.

Proof: Assume that $B$ is left–inverse for $A$ and assume that $\begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix}$ is a solution of the equation

$$c_1 Ae_1 + c_2 Ae_2 + \cdots + c_n Ae_n = 0$$

in $\mathbb{R}^m$. Then, by the associative and distributive property of matrix multiplication,

$$A(c_1 e_1 + c_2 e_2 + \cdots + c_n e_n) = 0.$$  

Multiplying on both sides by $B$ on the left we obtain, by the associative property of matrix multiplication,

$$BA(c_1 e_1 + c_2 e_2 + \cdots + c_n e_n) = B0,$$

or

$$c_1 e_1 + c_2 e_2 + \cdots + c_n e_n = 0,$$

since $BA = I$. We therefore conclude that

$$c_1 = c_2 = \cdots = c_n = 0,$$

since $\{e_1, e_2, \ldots, e_n\}$ is a basis for $\mathbb{R}^n$. Hence, the set $\{Ae_1, Ae_2, \ldots, Ae_n\}$ is a linearly independent. \hfill $\square$

(b) Prove that if $A$ has a right–inverse, $C$, then the set $\{Ae_1, Ae_2, \ldots, Ae_n\}$ spans $\mathbb{R}^m$.

Proof: Let $C$ denote a right–inverse of $A$. Then, for any vector, $b$, in $\mathbb{R}^m$, the equation

$$Ax = b$$

has a solution in $\mathbb{R}^n$ given by $x = Cb$. 

Write \( x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = x_1e_1 + x_2e_2 + \cdots + x_ne_n, \) so that

\[
\begin{align*}
b &= Ax \\
&= A(x_1e_1 + x_2e_2 + \cdots + x_ne_n) \\
&= x_1Ae_1 + x_2Ae_2 + \cdots + x_nAe_n,
\end{align*}
\]

where we have used the distributive property of matrix multiplication. We therefore see that \( b \in \text{span}\{Ae_1, Ae_2, \ldots, Ae_n\}. \)

2. Assume \( A \in \mathbb{M}(n, n) \) is invertible. Prove that the columns of \( A \) form a basis for \( \mathbb{R}^n \).

Proof: Assume that \( A \) is invertible. Then, \( A \) has both a left inverse and a right inverse. Observe that \( Ae_1, Ae_2, \ldots, Ae_n \) are the columns of \( A \). Consequently, by the result of Problem (1), the set \( \{Ae_1, Ae_2, \ldots, Ae_n\} \) is linearly independent and spans \( \mathbb{R}^n \). Hence, the columns of \( A \) form a basis for \( \mathbb{R}^n \).

3. Let \( A \) and \( B \) denote \( n \times n \) matrices. Prove that if \( A \) and \( B \) are invertible, then so is their product, \( AB \), and compute \( (AB)^{-1} \) in terms of \( A^{-1} \) and \( B^{-1} \).

Proof: Assume that \( A \) and \( B \) are invertible \( n \times n \) matrices with inverses \( A^{-1} \) and \( B^{-1} \), respectively. Observe that, by the associative property of matrix multiplication,

\[
(B^{-1}A^{-1})(AB) = B^{-1}(A^{-1}A)B = B^{-1}IB = B^{-1}B = I
\]

and

\[
(AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1} = AIA^{-1} = AA^{-1} = I.
\]

Hence, \( AB \) is invertible and

\[
(AB)^{-1} = B^{-1}A^{-1}.
\]
4. An \( n \times n \) matrix, \( E \), is said to be an **elementary matrix** if it is the result of performing an elementary row operation on the \( n \times n \) identity matrix, \( I \). Consider the following \( 3 \times 3 \) matrices

\[
E_1 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad E_2 = \begin{pmatrix} 1 & 0 & 0 \\ c & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \text{and} \quad E_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & d \end{pmatrix},
\]

where \( c \) and \( d \) are scalars with \( d \neq 0 \).

(a) Explain why \( E_1, E_2 \) and \( E_3 \) are elementary matrices.

\textbf{Solution:} Observe that \( E_1, E_2 \) and \( E_3 \) are obtained by performing the elementary row operations \( R_1 \leftrightarrow R_3 \), \( cR_1 + R_2 \rightarrow R_2 \) and \( dR_3 \rightarrow R_3 \), respectively, of the \( 3 \times 3 \) identity matrix. \( \square \)

(b) Show that \( E_1, E_2 \) and \( E_3 \) are invertible and compute their inverses. Are the inverses also elementary matrices?

\textbf{Solution:} Consider the elementary matrices

\[
F_1 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad F_2 = \begin{pmatrix} 1 & 0 & 0 \\ -c & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \text{and} \quad F_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/d \end{pmatrix},
\]

and compute

\[
F_1E_1 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = I.
\]

Observe that the calculation also shows that \( E_1F_1 = I \).

Similarly,

\[
F_2E_2 = \begin{pmatrix} 1 & 0 & 0 \\ -c & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ c & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = I,
\]

\[
E_2F_2 = \begin{pmatrix} 1 & 0 & 0 \\ c & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ -c & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = I,
\]

\[
F_3E_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/d \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & d \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = I,
\]
and
\[
E_3 F_3 = \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & d
\end{pmatrix} \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1/d
\end{pmatrix} = \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix} = I.
\]

\(\square\)

(c) Given an \(3 \times 3\) matrix \(A\), what is the result of multiplying \(A\) by \(E_1\), \(E_2\) and \(E_3\) on the left; that is, what are \(E_iA\) for \(i = 1, 2, 3\)?

**Solution:** Let \(A\) denote any \(3 \times 3\) matrix and write
\[
A = \begin{pmatrix}
a_{11} & a_{12} & a_{13} \\
a_{21} & a_{22} & a_{23} \\
a_{31} & a_{32} & a_{33}
\end{pmatrix}.
\]

Then,
\[
E_1 A = \begin{pmatrix}
0 & 0 & 1 \\
0 & 1 & 0 \\
1 & 0 & 0
\end{pmatrix} \begin{pmatrix}
a_{11} & a_{12} & a_{13} \\
a_{21} & a_{22} & a_{23} \\
a_{31} & a_{32} & a_{33}
\end{pmatrix} = \begin{pmatrix}
a_{31} & a_{32} & a_{33} \\
a_{21} & a_{22} & a_{23} \\
a_{11} & a_{12} & a_{13}
\end{pmatrix} = \begin{pmatrix} R_3 \\ R_2 \\ R_1 \end{pmatrix},
\]

where \(R_1\), \(R_2\) and \(R_3\) denote the rows of \(A\). Hence, the effect of multiplying \(A\) by \(E\) on the left is to perform the elementary row operation \(R_1 \leftrightarrow R_3\) on \(A\), which was the same elementary row operation that was used on \(I\) to obtain \(E_1\).
Next, compute
\[
E_2 A = \begin{pmatrix} 1 & 0 & 0 \\ c & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}
\]
\[
= \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ ca_{11} + a_{21} & ca_{12} + a_{22} & ca_{13} + a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}
\]
\[
= \begin{pmatrix} R_1 \\ cR_1 + R_2 \\ R_3 \end{pmatrix},
\]
which is the matrix \( A \) after the elementary row operation \( cR_1 + R_2 \leftrightarrow R_2 \) is performed. This is same elementary row operation that was used on \( I \) to obtain \( E_2 \).

Finally, compute
\[
E_3 A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & d \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}
\]
\[
= \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ da_{31} & da_{32} & da_{33} \end{pmatrix}
\]
\[
= \begin{pmatrix} R_1 \\ R_2 \\ dR_3 \end{pmatrix},
\]
which is the result of performing \( dR_3 \to R_3 \) on \( A \). This was the same of operation that led from \( I \) to \( E_3 \).

5. Let \( A \in \mathbb{M}(n, n) \) be invertible. Prove that the transpose, \( A^T \), of \( A \) is also invertible and compute its inverse. Deduce, therefore, that, if \( A \) is invertible, then the rows of \( A \) are linearly independent.

**Proof:** Assume that \( A \in \mathbb{M}(n, n) \) be invertible and let \( A^{-1} \) denote the inverse of \( A \). Then,
\[
A^{-1}A = AA^{-1} = I.
\]
Transposing all the terms of the previous equation we obtain

\[(A^{-1}A)^T = (AA^{-1})^T = I^T,\]

or

\[A^T(A^{-1})^T = (A^{-1})^TA^T = I,\]

which shows that \((A^{-1})^T\) is a left and right inverse for \(A^T\). Therefore, \(A^T\) is invertible and

\[(A^T)^{-1} = (A^{-1})^T;\]

that is, the inverse of \(A^T\) is the transpose of the inverse of \(A\).

Now, the columns of \(A^T\) are the rows of \(A\). Hence, by Problem (2), if \(A\) is invertible, the columns of \(A^T\) are linearly independent, since we have just shown that \(A^T\) is also invertible. Hence, the rows of \(A\) are linearly independent. \(\square\)