Assignment #18

Due on Monday, April 6, 2009

Read Section 5.2 on *Inverses* in Messer (pp. 184–190).

Background and Definitions

Given an $m \times n$ matrix, A,

The **null space of** A, denoted by $\mathcal{N}_A$, is the solution space of the homogeneous system

$$Ax = \mathbf{0}.$$

Thus, $\mathcal{N}_A$ is a subspace of $\mathbb{R}^n$. The dimension of $\mathcal{N}_A$ is called the **nullity** of A and is denoted by $n(A)$.

The **column space of** A, denoted by $\mathcal{C}_A$, is the span of the columns of A. It is therefore a subspace of $\mathbb{R}^m$. Its dimension is called the **rank** of A and is denoted by $r(A)$.

The **row space of** A, denoted by $\mathcal{R}_A$, is the subspace of $\mathbb{M}(1, n)$ spanned by the rows of A. Its dimension is the same as the rank of A, $r(A)$.

Do the following problems

1. Let $A = \begin{pmatrix} 1 & -4 & 1 \\ 0 & 3 & -1 \\ -3 & 0 & 1 \end{pmatrix}$. Compute the nullity and rank of A and verify that

$$n(A) + r(A) = 3.$$

2. Let $A \in \mathbb{M}(m, n)$.
   (a) Prove that $n(A) = 0$ if and only if the columns of A are linearly independent.
   (b) Prove that $r(A) = m$ if and only if the columns of A span $\mathbb{R}^m$.

3. Let $A \in \mathbb{M}(n, n)$. Prove that $n(A) = 0$ if and only if A invertible.

4. In the text for this course, on page 188, Messer defines the rank of a matrix $A \in \mathbb{M}(m,n)$ to be the number of leading 1s in the reduced row–echelon form of the matrix. Prove that this definition is equivalent to the one given in this assignment; that is, prove that the number of leading 1s in the reduced row–echelon form A is the dimension of the column space of A.

5. Let $A \in \mathbb{M}(m,n)$ and write $A = \begin{pmatrix} R_1 \\ R_2 \\ \vdots \\ R_m \end{pmatrix}$, where $R_1, R_2, \ldots, R_m$ denote the rows of A. Define $\mathcal{R}_A^\perp$ to be the set

$$\mathcal{R}_A^\perp = \{w \in \mathbb{R}^n \mid R_i w = 0 \text{ for all } i = 1, 2, \ldots, m\};$$

that is, $\mathcal{R}_A^\perp$ is the set of vectors in $\mathbb{R}^n$ which are orthogonal to the vectors $R_1^T, R_2^T, \ldots, R_m^T$ in $\mathbb{R}^n$.

(a) Prove that $\mathcal{R}_A^\perp$ is a subspace of $\mathbb{R}^n$.

(b) Prove that $\mathcal{R}_A^\perp = \mathcal{N}_A$.

(c) Let v denote a vector in $\mathbb{R}^n$. Prove that if $v \in \mathcal{N}_A$ and $v^T \in \mathcal{R}_A$, then $v = \mathbf{0}$.