Assignment #19

Due on Monday, April 13, 2009

Read Section 6.1 on *Linear Functions* in Messer (pp. 212–216).

Background and Definitions

Linear Functions. A function $T\colon \mathbb{R}^n \to \mathbb{R}^m$ is said to be **linear** if

(i) $T(cv) = cT(v)$ for all scalars c and all $v \in \mathbb{R}^n$, and

(ii) $T(u + v) = T(u) + T(v)$ for all $u, v \in \mathbb{R}^n$.

Do the following problems

1. Define $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ as follows: For each $v \in \mathbb{R}^2$, $T(v)$ is the reflection of the point determined by the coordinates of v, relative to the standard basis in $\mathbb{R}^2$, on the line $y = x$ in $\mathbb{R}^2$. That is, $T(v)$ determines a point along a line through the point determined by v which is perpendicular to the line $y = x$, and the distance from v to the line $y = x$ is the same as the distance from $T(v)$ to the line $y = x$ (see Figure 1).

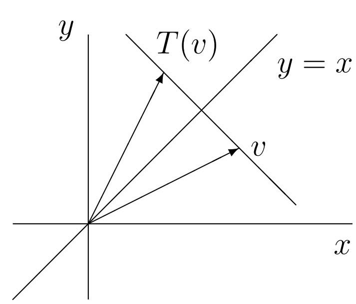

Figure 1: Reflection on the line $y = x$

Prove that T is a linear function.

2. Prove that if $T\colon \mathbb{R}^n \to \mathbb{R}^m$ is linear, then $T(\mathbf{0}) = \mathbf{0}$, where the first $\mathbf{0}$ denotes the zero–vector in $\mathbb{R}^n$ and the second $\mathbf{0}$ denotes the zero–vector in $\mathbb{R}^m$.

3. Suppose that $T\colon \mathbb{R}^n \to \mathbb{R}^m$ is linear and define

$$\mathcal{N}_T = \{v \in \mathbb{R}^n \mid T(v) = \mathbf{0}\},$$

   where $\mathbf{0}$ denotes the zero–vector in $\mathbb{R}^m$.

   Prove that $\mathcal{N}_T$ is a subspace of $\mathbb{R}^n$.

   Note: $\mathcal{N}_T$ is called the **null space** of the linear function T.

4. Suppose that $T\colon \mathbb{R}^n \to \mathbb{R}^m$ is linear and define

$$\mathcal{I}_T = \{w \in \mathbb{R}^m \mid w = T(v) \text{ for some } v \in \mathbb{R}^n\}.$$

   Prove that $\mathcal{I}_T$ is a subspace of $\mathbb{R}^m$.

   Note: The set $\mathcal{I}_T$ is called the **image** of the function T. It is also denoted by $T(\mathbb{R}^n)$; thus,

$$T(\mathbb{R}^n) = \{w \in \mathbb{R}^m \mid w = T(v) \text{ for some } v \in \mathbb{R}^n\}.$$

5. Fix $u \in \mathbb{R}^n$ and define $f\colon \mathbb{R}^n \to \mathbb{R}$ by

$$f(v) = \langle u, v\rangle \quad \text{for all } v \in \mathbb{R}^n.$$

   (a) Prove that f is a linear function.

   (b) Let $\mathcal{N}_f$ denote the null space of f; that is,

$$\mathcal{N}_f = \{v \in \mathbb{R}^n \mid \langle u, v\rangle = 0\}.$$

   Find the dimension of $\mathcal{N}_f$ for each of the cases: $u = \mathbf{0}$ and $u \neq \mathbf{0}$.