Solutions to Assignment #21

1. Given two vector–valued functions, $T$ and $R$, from $\mathbb{R}^n$ to $\mathbb{R}^m$, we can define the sum, $T + R$, of $T$ and $R$ by

$$(T + R)(v) = T(v) + R(v) \quad \text{for all} \quad v \in \mathbb{R}^n.$$ 

(a) Verify that, if both $T$ and $R$ are linear, then so is $T + R$.

**Solution:** We need to verify that

(i) $(T + R)(cv) = c(T + R)(v)$ for all $v \in \mathbb{R}^n$ and all scalars $c$,

and

(ii) $(T + R)(v + w) = (T + R)(v) + (T + R)(w)$ for all $v, w \in \mathbb{R}^n$.

To verify (i), compute

$$(T + R)(cv) = T(cv) + R(cv) = cT(v) + cR(v),$$

since $T$ and $R$ are linear. It then follows that

$$(T + R)(cv) = c(T(v) + R(v)) = c(T + R)(v),$$

which shows (i).

Next, compute

$$(T + R)(v + w) = T(v + w) + R(v + w) = T(v) + T(w) + R(v) + R(w),$$

since $T$ and $R$ are linear. Using the commutative and associative properties of vector addition we then get that

$$(T + R)(v + w) = (T(v) + R(v)) + (T(w) + R(w)) = (T + R)(v) + (T + R)(w),$$

which is (ii). □

(b) Explain how to define the scalar multiple $aT : \mathbb{R}^n \to \mathbb{R}^m$ of a vector valued function, $T : \mathbb{R}^n \to \mathbb{R}^m$, where $a$ is a scalar and verify that if $T$ is linear then so is $aT$.

**Solution:** Define $aT : \mathbb{R}^n \to \mathbb{R}^m$ by

$$(aT)(v) = a(T(v)) \quad \text{for all} \quad v \in \mathbb{R}^n.$$ 

We verify that
(i) \((aT)(cv) = c(aT)(v)\) for all \(v \in \mathbb{R}^n\) and all scalars \(c\), and
(ii) \((aT)(v + w) = (aT)(v) + (aT)(w)\) for all \(v, w \in \mathbb{R}^n\).

To verify (i) compute
\[
(aT)(cv) = a(T(cv)) = a(cT(v)),
\]
since \(T\) is linear; therefore, by the associativity and commutativity of multiplication of real numbers,
\[
(aT)(cv) = (ac)T(v) = (ca)T(v) = c(aT(v)) = c(aT)(v),
\]
which verifies (i).

To verify (ii), compute
\[
(aT)(v + w) = a(T(v + w)) = a(T(v) + T(w)),
\]
since \(T\) is linear. Thus, by the distributive property,
\[
(aT)(v + w) = a(T(v)) + a(T(w)) = (aT)(v) + (aT)(w),
\]
which is (ii).

\(\square\)

2. The identity function, \(I : \mathbb{R}^n \rightarrow \mathbb{R}^n\), is defined by
\[
I(v) = v\quad\text{for all } v \in \mathbb{R}^n.
\]

(a) Verify that \(I : \mathbb{R}^n \rightarrow \mathbb{R}^n\) is a linear transformation.

\textbf{Solution:} Compute
\[
I(cv) = cv = cI(v)
\]
and
\[
I(v + w) = v + w = I(v) + I(w).
\]

\(\square\)

(b) Give the matrix representation of \(I\) relative to the standard basis in \(\mathbb{R}^n\).

\textbf{Solution:} Compute \(I(e_j) = e_j\) for \(j = 1, 2, \ldots, n\). Then,
\[
M_I = [I(e_1) \quad I(e_2) \quad \cdots \quad I(e_n)]
= [e_1 \quad e_2 \quad \cdots \quad e_n]
= I,
\]
where the last \(I\) denotes the \(n \times n\) identity matrix. Thus, the matrix representation of the identity function if the identity matrix.

\(\square\)
(c) Compute the null space, $N_I$, and image, $I_I$, of $I$.

**Solution:** Note that if $v$ is a solution of $I(v) = 0$, then $v = 0$. It then follows that

$$N_I = \{0\}.$$ 

Observe that for every $w \in \mathbb{R}^n$, $w = I(w)$. It then follows that

$$I_I = \mathbb{R}^n.$$ 

□

3. The zero function, $O : \mathbb{R}^n \to \mathbb{R}^m$, is defined by

$$O(v) = 0 \quad \text{for all} \quad v \in \mathbb{R}^n.$$ 

(a) Verify that $O : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation.

**Solution:** Compute

$$C(cv) = 0 = cO = cO(v)$$

and

$$O(v + w) = 0 = 0 + 0 = O(v) + O(v).$$ 

□

(b) Give the matrix representation of $O$ relative to the standard bases in $\mathbb{R}^n$ and $\mathbb{R}^m$.

**Solution:** Compute $O(e_j) = 0$ for $j = 1, 2, \ldots n$. Then,

$$M_O = \begin{bmatrix} O(e_1) & O(e_2) & \cdots & O(e_n) \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & \cdots & 0 \end{bmatrix}$$

$$= O,$$

where the last $O$ denotes the $n \times n$ zero matrix. Thus, the matrix representation of the zero function if the zero matrix. □

(c) Compute the null space, $N_O$, and image, $I_O$, of $O$. 


Solution: Note that $O(v) = 0$, for all $v \in \mathbb{R}^n$; thus,
\[ \mathcal{N}_O = \mathbb{R}^n. \]
Since $O(v) = 0$ for all $v \in \mathbb{R}^n$, every vector in $\mathbb{R}^n$ gets mapped to $0$. Therefore,
\[ \mathcal{I}_O = \{0\}. \]
\[ \square \]

4. Let $T: \mathbb{R}^n \to \mathbb{R}^m$ denote a linear function and let $M_T \in \mathbb{M}(m, n)$ be its matrix representation with respect to the standard bases in $\mathbb{R}^n$ and $\mathbb{R}^m$.

(a) Prove that the null space of $T$, $\mathcal{N}_T$, is the null space of the matrix $M_T$.

Solution: Observe that
\[ v \in \mathcal{N}_T \quad \text{iff} \quad T(v) = 0 \]
\[ \quad \text{iff} \quad M_T v = 0 \]
\[ \quad \text{iff} \quad v \in \mathcal{N}_{M_T}. \]
Thus, $\mathcal{N}_T = \mathcal{N}_{M_T}$. \[ \square \]

(b) Prove that the image of $T$, $\mathcal{I}_T$, is the span of the columns of the matrix $M_T$.

Solution: Observe that
\[ w \in \mathcal{I}_T \quad \text{iff} \quad w = T(v) \text{ for some } v \in \mathbb{R}^n \]
\[ \quad \text{iff} \quad w = M_T v \]
\[ \quad \text{iff} \quad w \in \text{span}\{M_T e_1, M_T e_2, \ldots, M_T e_n\}. \]
Thus, $\mathcal{I}_T$ is the span of the columns of $M_T$. \[ \square \]

5. If $T: \mathbb{R}^n \to \mathbb{R}^n$ is a function, we can define the iterates, $T^k$, of $T$, where $k$ is a positive integer, as follows:
\[ T^2 = T \circ T; \]
That is, $T$ is the composition of $T$ with itself. Next, define
\[ T^3 = T^2 \circ T \]
and so on. More precisely, once we have defined $T^{k-1}$ for $k > 1$, we can define $T^k$ by
\[ T^k = T^{k-1} \circ T. \]
(a) Prove that if $T$ is a linear function from $\mathbb{R}^n$ to $\mathbb{R}^n$, then so are the functions $T^k$ for $k = 1, 2, \ldots$.

\textbf{Solution}: This result follows from the fact that compositions of linear functions are linear. □

(b) Prove that $T^m$ and $T^k$ commute with each other; that is,

$$T^m \circ T^k = T^k \circ T^m,$$

where $k$ and $m$ are positive integers.

\textbf{Solution}: By the associativity of composition we have that

$$T^m \circ T^k = T^{m+k} = T^{k+m} = T^k \circ T^m.$$ □

(c) Given $v \in \mathbb{R}^n$, prove that the set

$$\{v, T(v), T^2(v), \ldots, T^n(v)\}$$

is linearly dependent.

\textbf{Solution}: Note that $\{v, T(v), T^2(v), \ldots, T^n(v)\}$ is subset of $\mathbb{R}^n$ with $n + 1$ elements. Thus, since $\dim(\mathbb{R}^n) = n$, it follows that $\{v, T(v), T^2(v), \ldots, T^n(v)\}$ is linearly dependent. □