## Assignment #24

**Due on Wednesday, April 29, 2009**

**Read** Section 7.2 on *Definition* of the Determinant in Messer (pp. 273–276).

**Read** Section 8.1 on *Definitions* of the eigenvalues and eigenvectors in Messer (pp. 303–307).

### Background and Definitions

**Eigenvalues and Eigenvectors.** Let $T\colon \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation. A scalar, $\lambda$, is said to be an **eigenvalue** of $T$ if and only if the equation

$$T(v) = \lambda v \tag{1}$$

has a nontrivial solution.

A nontrivial solution, $v$, of the equation $T(v) = \lambda v$ is called an **eigenvector** corresponding to the eigenvalue $\lambda$.

Observe that the equation in (1) can also be written as

$$(T - \lambda I)v = \mathbf{0},$$

where $I\colon \mathbb{R}^n \to \mathbb{R}^n$ denotes the identity transformation in $\mathbb{R}^n$. Thus, $\lambda$ is an eigenvalue of $T$ if and only if the null space of the linear transformation $T - \lambda I$ is nontrivial; that is $\mathcal{N}_{T-\lambda I} \neq \{\mathbf{0}\}$. The null space of $T - \lambda I$ is called the **eigenspace** of $T$ corresponding to $\lambda$ and is denoted by $E_T(\lambda)$.

**Do** the following problems

1. Let $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ denote a linear transformation in $\mathbb{R}^2$. Suppose that $v_1$ and $v_2$ are two eigenvectors of $T$ corresponding to the eigenvalues $\lambda_1$ and $\lambda_2$, respectively.

   Prove that, if $\lambda_1 \neq \lambda_2$, then the set $\{v_1, v_2\}$ is linearly independent.

   Deduce therefore that a linear transformation, $T$, from $\mathbb{R}^2$ to $\mathbb{R}^2$ cannot have more than two distinct eigenvalues.

2. Show that the rotation $R_\theta\colon \mathbb{R}^2 \to \mathbb{R}^2$ does not have any real eigenvalues unless $\theta = 0$ or $\theta = \pi$.

   Give the eigenvalues and corresponding eigenspaces in each case.

3. Let $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ be given by

$$T\begin{pmatrix} x \\ y \end{pmatrix} = A\begin{pmatrix} x \\ y \end{pmatrix}, \quad \text{for all} \quad \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2,$$

where $A$ is the $2 \times 2$ matrix

$$A = \begin{pmatrix} 5 & 3 \\ -6 & -4 \end{pmatrix}.$$

Find all the eigenvalue of $T$ and compute their respective eigenspaces.

4. Let $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ be given by

$$T\begin{pmatrix} x \\ y \end{pmatrix} = A\begin{pmatrix} x \\ y \end{pmatrix}, \quad \text{for all} \quad \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2,$$

where $A$ is the $2 \times 2$ matrix

$$A = \begin{pmatrix} a & b \\ b & a \end{pmatrix},$$

where $a$ and $b$ are real constants.

(a) Show that $T$ has real eigenvalues.

(b) Under what conditions on $a$ and $b$ will the eigenvalues obtained in part (a) be distinct eigenvalues?

5. Let $T\colon \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation. Prove that $\lambda = 0$ is an eigenvalue of $T$ if and only if the matrix representation, $M_T$, of $T$ is singular.