1. Let $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ denote a linear transformation in $\mathbb{R}^2$. Suppose that $v_1$ and $v_2$ are two eigenvectors of $T$ corresponding to the eigenvalues $\lambda_1$ and $\lambda_2$, respectively.

Prove that, if $\lambda_1 \neq \lambda_2$, then the set $\{v_1, v_2\}$ is linearly independent.

Deduce therefore that a linear transformation, $T$, from $\mathbb{R}^2$ to $\mathbb{R}^2$ cannot have more than two distinct eigenvalues.

Proof: Let $\lambda_1$ and $\lambda_2$ be distinct eigenvalues of $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ with corresponding eigenvectors $v_1$ and $v_2$, respectively.

Suppose that $c_1$ and $c_2$ solve the vector equation

$$c_1v_1 + c_2v_2 = 0. \quad (1)$$

Applying $T$ on both sides of the equation in (1) and using the linearity of $T$, we obtain that

$$c_1T(v_1) + c_2T(v_2) = 0,$$

or

$$c_1\lambda_1v_1 + c_2\lambda_2v_2 = 0, \quad (2)$$

since $v_1$ and $v_2$ are eigenvectors corresponding to $\lambda_1$ and $\lambda_2$, respectively.

Since, we are assuming that $\lambda_1$ and $\lambda_2$ are distinct, one of them cannot be 0. Thus, suppose that $\lambda_2 \neq 0$ and multiply the vector equation in (1) by $\lambda_2$ to get

$$c_1\lambda_2v_1 + c_2\lambda_2^2v_2 = 0. \quad (3)$$

Subtracting the vector equation in (1) from the vector equation in (3) we then get that

$$c_1(\lambda_2 - \lambda_1)v_1 = 0,$$

which implies that

$$c_1v_1 = 0$$

because $\lambda_1 \neq \lambda_2$. It then follows that $c_1 = 0$ since $v_1$ is not the zero vector in $\mathbb{R}^2$. We then get from (1) that

$$c_2v_2 = 0,$$

which implies that $c_2 = 0$ since $v_2$ is not the zero vector in $\mathbb{R}^2$. 


We have therefore shown that \( c_1 = c_2 = 0 \) is the only solution of the vector equation in (1). Consequently, \( \{v_1, v_2\} \) is linearly independent.

Thus, a linear transformation, \( T \), from \( \mathbb{R}^2 \) to \( \mathbb{R}^2 \) cannot have more than two distinct eigenvalues. For if it did, then a similar argument to the one given above would imply that there is a set of more than two linearly independent vectors, which is impossible in \( \mathbb{R}^2 \) because \( \text{dim}(\mathbb{R}^2) = 2 \).

2. Show that the rotation \( R_\theta : \mathbb{R}^2 \to \mathbb{R}^2 \) does not have any real eigenvalues unless \( \theta = 0 \) or \( \theta = \pi \).

Give the eigenvalues and corresponding eigenspaces in each case.

**Solution:** Consider the matrix for the transformation \( R_\theta - \lambda I \), where \( I \) denotes the \( 2 \times 2 \) identity matrix,

\[
M_{R_\theta - \lambda I} = \begin{pmatrix} \cos \theta - \lambda & -\sin \theta \\ \sin \theta & \cos \theta - \lambda \end{pmatrix}.
\]

The homogeneous system \((R_\theta - \lambda I)v = 0\) has nontrivial solutions if and only if the columns of \( M_{R_\theta - \lambda I} \) are linearly dependent, and this is the case if and only if \( \det(M_{R_\theta - \lambda I}) = 0 \), or

\[
(cos \theta - \lambda)^2 + \sin^2 \theta = 0,
\]

which implies that

\[
\lambda^2 - 2 \cos \theta \lambda + 1 = 0. \tag{4}
\]

The quadratic equation in (4) has real solutions if and only if

\[
4 \cos^2 \theta - 4 \geq 0,
\]

or

\[
\cos^2 \theta \geq 1.
\]

But, \( \cos^2 \theta \leq 1 \). We therefore get that

\[
\cos^2 \theta = 1.
\]

Thus, either \( \cos \theta = 1 \), which yields \( \theta = 0 \) or \( 2\pi \), or \( \cos \theta = -1 \), which yields \( \theta = \pi \) or \( -\pi \). \( \square \)
3. Let $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be given by

$$T \begin{pmatrix} x \\ y \end{pmatrix} = A \begin{pmatrix} x \\ y \end{pmatrix}, \quad \text{for all} \quad \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2,$$

where $A$ is the $2 \times 2$ matrix

$$A = \begin{pmatrix} 5 & 3 \\ -6 & -4 \end{pmatrix}.$$

Find all the eigenvalues of $T$ and compute their respective eigenspaces.

**Solution:** We look for values of $\lambda$ for which the equation

$$T(v) = \lambda v$$

has nontrivial solutions. This is equivalent to finding values of $\lambda$ for which the homogeneous system

$$(A - \lambda I)v = 0 \quad (5)$$

has nontrivial solutions. The system in (5) has nontrivial solutions when the columns of the matrix

$$A - \lambda I = \begin{pmatrix} 5 - \lambda & 3 \\ -6 & -4 - \lambda \end{pmatrix}$$

are linearly dependent. This happens when $\det(A\lambda I) = 0$, or

$$(\lambda - 5)(\lambda + 4) + 18 = 0. \quad (6)$$

Solving the equation (6) for $\lambda$ yields the values $\lambda_1 = -1$ and $\lambda_2 = 2$. These are the eigenvalues of $T$.

To find $E_T(-1)$ we solve the homogeneous system

$$(A + I)v = 0,$$

or

$$\begin{pmatrix} 6 & 3 \\ -6 & -3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad (7)$$

Using Gaussian elimination we see that the system in (7) is equivalent to the equation

$$x + \frac{1}{2}y = 0,$$
which has solution space given by
\[
\begin{pmatrix} x \\ y \end{pmatrix} = t \begin{pmatrix} 1 \\ -2 \end{pmatrix}.
\]

It then follows that
\[
E_T(-1) = \text{span}\left\{ \begin{pmatrix} 1 \\ -2 \end{pmatrix} \right\}.
\]

Similar calculations show that
\[
E_T(2) = \text{span}\left\{ \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right\}.
\]

\[\square\]

4. Let \( T: \mathbb{R}^2 \to \mathbb{R}^2 \) be given by
\[
T \begin{pmatrix} x \\ y \end{pmatrix} = A \begin{pmatrix} x \\ y \end{pmatrix}, \quad \text{for all } \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2,
\]
where \( A \) is the \( 2 \times 2 \) matrix
\[
A = \begin{pmatrix} a & b \\ b & a \end{pmatrix},
\]
where \( a \) and \( b \) are real constants.

(a) Show that \( T \) has real eigenvalues.

\textbf{Solution:} We look for values of \( \lambda \) for which the system
\[
(A - \lambda I)v = 0 \quad \text{(8)}
\]
has nontrivial solutions. This happens when the columns of the matrix
\[
A - \lambda I = \begin{pmatrix} a - \lambda & b \\ b & a - \lambda \end{pmatrix}
\]
are linearly dependent. Thus, we require that \( \det(A - \lambda I) = 0 \), or
\[
(\lambda - a)^2 - b^2 = 0. \quad \text{(9)}
\]
We can factor the equation in (9) to get
\[
(\lambda - a + b)(\lambda - a - b) = 0,
\]
which yields the values \( \lambda_1 = a - b \) and \( \lambda_2 = a + b \), both of which are real. \[\square\]
(b) Under what conditions on \( a \) and \( b \) will the eigenvalues obtained in part (a) be distinct eigenvalues?

**Solution:** \( \lambda_1 = \lambda_2 \) if and only if and only if \( a - b = a + b \), which occurs if and only if \( b = 0 \). \( \square \)

5. Let \( T : \mathbb{R}^n \to \mathbb{R}^n \) be a linear transformation. Prove that \( \lambda = 0 \) is an eigenvalue of \( T \) if and only if the matrix representation, \( M_T \), of \( T \) is singular.

**Solution:** \( \lambda = 0 \) is an eigenvalue of \( T \) if and only if the equation

\[
T(v) = 0, \quad v,
\]

or

\[
T(v) = 0,
\]

has a nontrivial solution. Thus, \( \lambda = 0 \) is an eigenvalue of \( T \) if and only if

\[
M_Tv = 0
\]

has nontrivial solutions, which is true if and only if \( M_T \) is singular, which is equivalent to \( T \) being singular. \( \square \)