Solutions to Assignment #4

1. Let \( S = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 \mid x \geq 0, y \geq 0 \right\} \). Show that \( S \) is closed under vector addition in \( \mathbb{R}^2 \). Explain why \( S \) is not a subspace of \( \mathbb{R}^2 \).

**Solution:** Let \( v = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} \) and \( w = \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} \) be vectors in \( S \). It then follows that \( x_1, y_1, x_2, y_2 \geq 0 \). Consequently,

\[
x_1 + x_2 \geq 0 \quad \text{and} \quad y_1 + y_2 \geq 0,
\]

which shows that

\[
v + w = \begin{pmatrix} x_1 + x_2 \\ y_1 + y_2 \end{pmatrix} \in S,
\]

and therefore \( S \) is closed under vector addition in \( \mathbb{R}^2 \). However, \( S \) is not a subspace of \( \mathbb{R}^2 \) because \( S \) is not closed under scalar multiplication; to see this, note that \( \begin{pmatrix} 1 \\ 1 \end{pmatrix} \in S, \) but

\[
(-1) \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \end{pmatrix} \notin S.
\]

\[\Box\]

2. Let \( a_1, a_2, b_1, b_2, c_1, c_2 \) be real constants. Let \( W \) be the solution set of the homogeneous system

\[
\begin{align*}
a_1x_1 + b_1x_2 + c_1x_3 &= 0 \\
a_2x_1 + b_2x_2 + c_2x_3 &= 0.
\end{align*}
\]

Prove that \( W \) is a subspace of \( \mathbb{R}^3 \).

**Solution:** Note that \( W \) is a subset of \( \mathbb{R}^3 \) given by

\[
W = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in \mathbb{R}^3 \mid \begin{cases} a_1x_1 + b_1x_2 + c_1x_3 = 0 \\ a_2x_1 + b_2x_2 + c_2x_3 = 0 \end{cases} \right\}
\]

First, observe that \( x_1 = x_2 = x_3 = 0 \) solves the system. Consequently, \( W \) is not empty.
Suppose that \[
\begin{pmatrix}
  x_1 \\
x_2 \\
x_3
\end{pmatrix}
\quad \text{and} \quad
\begin{pmatrix}
  y_1 \\
y_2 \\
y_3
\end{pmatrix}
\]
are solutions of the system. Then,
\[
\begin{align*}
  a_1 x_1 + b_1 x_2 + c_1 x_3 &= 0 \\
  a_2 x_1 + b_2 x_2 + c_2 x_3 &= 0,
\end{align*}
\]
and
\[
\begin{align*}
  a_1 y_1 + b_1 y_2 + c_1 y_3 &= 0 \\
  a_2 y_1 + b_2 y_2 + c_2 y_3 &= 0.
\end{align*}
\]
Adding the first equations of the systems and the second equations yields
\[
\begin{align*}
  a_1 (x_1 + y_1) + b_1 (x_2 + y_2) + c_1 (x_3 + y_3) &= 0 \\
  a_2 (x_1 + y_1) + b_2 (x_2 + y_2) + c_2 (x_3 + y_3) &= 0,
\end{align*}
\]
where we have used the distributive property for real numbers. It then follows that \[
\begin{pmatrix}
  x_1 + y_1 \\
x_2 + y_2 \\
x_3 + y_3
\end{pmatrix}
\]
is a solution of the system, and therefore \( W \) is closed under vector addition in \( \mathbb{R}^3 \).

Next, suppose that \[
\begin{pmatrix}
  x_1 \\
x_2 \\
x_3
\end{pmatrix}
\]
is a solution of the system
\[
\begin{align*}
  a_1 x_1 + b_1 x_2 + c_1 x_3 &= 0 \\
  a_2 x_1 + b_2 x_2 + c_2 x_3 &= 0.
\end{align*}
\]
Multiplying both equations in the system by a scalar \( t \) we obtain
\[
\begin{align*}
  a_1 (tx_1) + b_1 (tx_2) + c_1 (tx_3) &= 0 \\
  a_2 (tx_1) + b_2 (tx_2) + c_2 (tx_3) &= 0,
\end{align*}
\]
where we have applied the distributive and associative properties for real numbers. It then follows that \[
\begin{pmatrix}
  tx_1 \\
tx_2 \\
tx_3
\end{pmatrix}
\]
\( \in W \), and therefore \( W \) is also closed under scalar multiplication. Hence, we conclude that \( W \) is a subspace of \( \mathbb{R}^3 \). \( \square \)

3. Let \( L = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 \mid y = 2x + 1 \right\} \). Determine whether or not \( L \) is a subspace of \( \mathbb{R}^2 \).
Solution: $L$ is not a subspace of $\mathbb{R}^2$. To see this, note that the vector 
\[
\begin{pmatrix}
0 \\
1
\end{pmatrix}
\] 
is in $L$; however, the vector $(-1) \begin{pmatrix}
0 \\
1
\end{pmatrix} = \begin{pmatrix}
0 \\
-1
\end{pmatrix}$ is not in $L$ since $y = -1$ and $x = 0$ do not satisfy the equation $y = 2x + 1$. □

4. Let $W$ be a subspace of $\mathbb{R}^n$. Use the definition of subspace to prove the following statements.

(a) If $v \in W$, then $W$ must also contain the additive inverse of $v$.

Proof: Since $W$ is subspace of $\mathbb{R}^n$, it is closed under scalar multiplication. It then follows that, if $v \in W$, then $(-1)v \in W$; that is $-v \in W$. □

(b) $W$ contains the zero vector.

Proof: Since $W$ is a subspace, it is non–empty; therefore, it contains a vector $v$. By the previous part, $-v \in W$. Hence, since $W$ is closed under vector addition, $v + (-v) \in W$, which shows that $0 \in W$. □

5. Given two subsets $A$ and $B$ of $\mathbb{R}^n$, the intersection of $A$ and $B$, denoted by $A \cap B$, is the set which contains all vectors that are both in $A$ and $B$; in symbols, 

$$A \cap B = \{v \in \mathbb{R}^n \mid v \in A \text{ and } v \in B\}.$$ 

(a) Prove that $A \cap B \subseteq A$ and $A \cap B \subseteq B$.

Proof: If $x \in A \cap B$ then $x \in A$ and $x \in B$, by the definition of intersection. Thus, $x \in B$. We have therefore shown that 

$$x \in A \cap B \Rightarrow x \in A,$$

which shows that $A \cap B \subseteq A$.

A similar argument shows that $A \cap B \subseteq B$. □

(b) Prove that if $W_1$ and $W_2$ are two subspaces of $\mathbb{R}^n$, then the intersection $W_1 \cap W_2$ is a subspace of $\mathbb{R}^n$ which is contained in both $W_1$ and $W_2$. 
Proof: We first show that $W_1 \cap W_2$ is a subspace of $\mathbb{R}^n$.

Since $W_1$ and $W_2$ are subspace of $\mathbb{R}^n$, it follows from the result in part (b) of problem 4 in this assignment that $0 \in W_1$ and $0 \in W_2$. Consequently, $0 \in W_1 \cap W_2$, which shows that $W_1 \cap W_2$ is not empty.

Next, suppose that $v, w \in W_1 \cap W_2$. Then, $v \in W_1$ and $w \in W_1$ so that

$$v + w \in W_1$$

since $W_1$ is closed under vector addition. Similarly, we can show that

$$v + w \in W_2.$$  

It then follows that

$$v + w \in W_1 \cap W_2,$$

and therefore $W_1 \cap W_2$ is closed under vector addition.

Finally, if $v \in W_1 \cap W_2$ and $t \in \mathbb{R}$, we have that $v \in W_1$ and $v \in W_2$ and therefore

$$tv \in W_1 \quad \text{and} \quad tv \in W_2$$

since $W_1$ and $W_2$ are closed under scalar multiplication. It then follows that

$$tv \in W_1 \cap W_2,$$

which shows that $W_1 \cap W_2$ is closed under scalar multiplication.

We have shown that $W_1 \cap W_2$ is a non–empty subset of $\mathbb{R}^2$ which is closed under the vector space operations of $\mathbb{R}^n$; that is, $W_1 \cap W_2$ is a subspace of $\mathbb{R}^n$.

Applying part (a) in this problem we also conclude that $W_1 \cap W_2$ is a subspace of $\mathbb{R}^n$ which is contained in both $W_1$ and $W_2$. 

$\Box$