Assignment #5

Due on Monday, February 9, 2009

Read Section 1.8 on *Subspaces* in Messer (pp. 39–44).

Read Section 3.2 on *Span* in Messer (pp. 97–102).

Background and Definitions

(Spans). For any subset S of $\mathbb{R}^n$, $\text{span}(S)$ is the smallest subspace of $\mathbb{R}^n$ which contains S; that is,

(i) $\text{span}(S)$ is a subspace of $\mathbb{R}^n$;

(ii) $S \subseteq \text{span}(S)$; and

(iii) for any subspace, W, of $\mathbb{R}^n$ such that $S \subseteq W$, $\text{span}(S) \subseteq W$.

Do the following problems

1. Let S_1 and S_2 denote two subsets of $\mathbb{R}^n$ such that $S_1 \subseteq S_2$.
   (a) Prove that $\text{span}(S_1) \subseteq \text{span}(S_2)$.
   (b) Prove that if S_1 spans $\mathbb{R}^n$, then $\text{span}(S_2) = \mathbb{R}^n$.

2. Let $S = \{v_1, v_2, \ldots v_k\}$, where be $v_1, v_2, \ldots v_k$ are vectors in $\mathbb{R}^n$. The symbol $S\backslash\{v_j\}$ denotes the set S with v_j removed from the set, for $j \in \{1, 2, \ldots, k\}$.

   Suppose that $v_j \in \text{span}(S\backslash\{v_j\})$ for some j in $\{1, 2, \ldots, k\}$. Prove that
   $$\text{span}(S\backslash\{v_j\}) = \text{span}(S).$$

3. Suppose that W is a subspace of $\mathbb{R}^n$ and that $v_1, v_2, \ldots, v_k \in W$. Prove that
   $$\text{span}\{v_1, v_2, \ldots, v_k\} \subseteq W.$$

4. Let W be a subspace of $\mathbb{R}^n$. Prove that if the set $\{v, w\}$ spans W, then the set $\{v, v + w\}$ also spans W.

5. Let W be the solution set of the homogeneous system
   $$\begin{cases} -x_1 + 2x_2 - 3x_3 &= 0 \\ 2x_1 - x_2 + 4x_3 &= 0. \end{cases}$$
   Solve the system to determine W, and find a set, S, of vectors in $\mathbb{R}^3$ such that
   $$W = \text{span}(S).$$
   Deduce, therefore, that W is a subspace of $\mathbb{R}^3$.