Assignment #5

Due on Monday, February 9, 2009

Read Section 1.8 on Subspaces in Messer (pp. 39–44).
Read Section 3.2 on Span in Messer (pp. 97–102).

Background and Definitions

(Spans). For any subset $S$ of $\mathbb{R}^n$, span($S$) is the smallest subspace of $\mathbb{R}^n$ which contains $S$; that is,

(i) span($S$) is a subspace of $\mathbb{R}^n$;
(ii) $S \subseteq$ span($S$); and
(iii) for any subspace, $W$, of $\mathbb{R}^n$ such that $S \subseteq W$, span($S$) $\subseteq W$.

Do the following problems

1. Let $S_1$ and $S_2$ denote two subsets of $\mathbb{R}^n$ such that $S_1 \subseteq S_2$.
   (a) Prove that span($S_1$) $\subseteq$ span($S_2$).
   (b) Prove that if $S_1$ spans $\mathbb{R}^n$, then span($S_2$) = $\mathbb{R}^n$.

2. Let $S = \{v_1, v_2, \ldots, v_k\}$, where be $v_1, v_2, \ldots, v_k$ are vectors in $\mathbb{R}^n$. The symbol $S\backslash\{v_j\}$ denotes the set $S$ with $v_j$ removed from the set, for $j \in \{1, 2, \ldots, k\}$.
   Suppose that $v_j \in$ span($S\backslash\{v_j\}$) for some $j$ in $\{1, 2, \ldots, k\}$. Prove that span($S\backslash\{v_j\}$) = span($S$).

3. Suppose that $W$ is a subspace of $\mathbb{R}^n$ and that $v_1, v_2, \ldots, v_k \in W$. Prove that span$\{v_1, v_2, \ldots, v_k\} \subseteq W$.

4. Let $W$ be a subspace of $\mathbb{R}^n$. Prove that if the set $\{v, w\}$ spans $W$, then the set $\{v, v + w\}$ also spans $W$.

5. Let $W$ be the solution set of the homogeneous system

\[
\begin{align*}
-x_1 + 2x_2 - 3x_3 &= 0 \\
2x_1 - x_2 + 4x_3 &= 0.
\end{align*}
\]

Solve the system to determine $W$, and find a set, $S$, of vectors in $\mathbb{R}^3$ such that $W = \text{span}(S)$.
Deduce, therefore, that $W$ is a subspace of $\mathbb{R}^3$. 