1. Let $S_1$ and $S_2$ denote two subsets of $\mathbb{R}^n$ such that $S_1 \subseteq S_2$.

(a) Prove that span($S_1$) $\subseteq$ span($S_2$).

Proof: Since $S_2 \subseteq$ span($S_2$), it follows from $S_1 \subseteq S_2$ that

$$S_1 \subseteq \text{span}(S_2).$$

Thus, since span($S_2$) is a subspace and span($S_1$) is the smallest subspace of $\mathbb{R}^n$ which contains $S_1$, we have that

$$\text{span}(S_1) \subseteq \text{span}(S_2),$$

which was to be shown. \qed

(b) Prove that if $S_1$ spans $\mathbb{R}^n$, then span($S_2$) $= \mathbb{R}^n$.

Proof: Since span($S_1$) $= \mathbb{R}^n$, it follows from part (a) that

$$\mathbb{R}^n \subseteq \text{span}(S_2).$$

Moreover, span($S_2$) $\subseteq \mathbb{R}^n$, since span($S_2$) is a subspace of $\mathbb{R}^n$. We therefore conclude that span($S_2$) $= \mathbb{R}^n$. \qed

2. Let $S = \{v_1, v_2, \ldots, v_k\}$, where be $v_1, v_2, \ldots, v_k$ are vectors in $\mathbb{R}^n$. The symbol $S\setminus\{v_j\}$ denotes the set $S$ with $v_j$ removed from the set, for $j \in \{1, 2, \ldots, k\}$. Suppose that $v_j \in \text{span}(S\setminus\{v_j\})$ for some $j$ in $\{1, 2, \ldots, k\}$. Prove that

$$\text{span}(S\setminus\{v_j\}) = \text{span}(S).$$

Proof: Observe that $S\setminus\{v_j\} \subseteq S$. Consequently, by part (a) in Problem 1,

$$\text{span}(S\setminus\{v_j\}) \subseteq \text{span}(S).$$

It remains to show, therefore, that

$$\text{span}(S) \subseteq \text{span}(S\setminus\{v_j\}).$$

To show this, let $v \in \text{span}(S)$, then

$$v = c_1v_1 + c_2v_2 + \cdots + c_jv_j + \cdots + c_kv_k,$$

(1)
for some scalars \(c_1, c_2, \ldots, c_k\). Now, since \(v_j \in \text{span}(S\{v_j\})\), there exist scalars \(d_1, d_2, \ldots, d_{j-1}, d_{j+1}, \ldots, d_k\) such that

\[
v_j = d_1 v_1 + d_2 v_2 + \cdots + d_{j-1} v_{j-1} + d_{j+1} v_{j+1} \cdots + d_k v_k.
\]

Substituting for \(v_j\) in (1) and using the distributive properties, we then get that

\[
v = c_1 v_1 + \cdots + c_j (d_1 v_1 + \cdots + d_{j-1} v_{j-1} + d_{j+1} v_{j+1} \cdots + d_k v_k) + \cdots + c_k v_k
\]

which is a linear combination of vectors is \(S\{v_j\}\). It then follows that

\[
v \in \text{span}(S) \Rightarrow v \in \text{span}(S\{v_j\}),
\]

or

\[
\text{span}(S) \subseteq \text{span}(S\{v_j\}),
\]

which finishes the proof. \(\Box\)

3. Suppose that \(W\) is a subspace of \(\mathbb{R}^n\) and that \(v_1, v_2, \ldots, v_k \in W\). Prove that \(\text{span}\{v_1, v_2, \ldots, v_k\} \subseteq W\).

*Proof:* Put \(S = \{v_1, v_2, \ldots, v_k\}\); then \(S \subseteq W\), where \(W\) is a subspace of \(\mathbb{R}^n\). It then follows that

\[
\text{span}(S) \subseteq W,
\]

since \(\text{span}(S)\) is the smallest subspace of \(\mathbb{R}^n\) which contains \(S\). \(\Box\)

4. Let \(W\) be a subspace of \(\mathbb{R}^n\). Prove that if the set \(\{v, w\}\) spans \(W\), then the set \(\{v, v + w\}\) also spans \(W\).

*Proof:* Suppose that \(W = \text{span}\{v, w\}\). Then, \(W\) is a subspace which contains \(v\) and \(w\). In then follows from the closure of \(W\) with respect to vector addition that \(v + w \in W\). We then have that

\[
v, v + w \in W.
\]

Thus, by the result of problem 4,

\[
\text{span}\{v, v + w\} \subseteq W.
\]
Ont the other hand, since \( W = \text{span}\{v, w\} \), \( u \in W \) implies that
\[
   u = c_1 v + c_2 w,
\]
for some scalars \( c_1 \) and \( c_2 \). Consequently,
\[
   u = c_1 v + c_2 w + c_2 v - c_2 v
   = (c_1 - c_2) v + c_2 (w + v),
\]
which shows that \( u \in \text{span}\{v, v + w\} \); thus,
\[
   u \in W \Rightarrow u \in \text{span}\{v, v + w\},
\]
or
\[
   W \subseteq \text{span}\{v, v + w\}.
\]
Combining this with (2) yields that
\[
   W = \text{span}\{v, v + w\};
\]
that is, the set \( \{v, v + w\} \) spans \( W \).

5. Let \( W \) be the solution set of the homogeneous system
\[
   \begin{aligned}
   -x_1 + 2x_2 - 3x_3 &= 0 \\
   2x_1 - x_2 + 4x_3 &= 0.
   \end{aligned}
\]
Solve the system to determine \( W \), and find a set, \( S \), of vectors in \( \mathbb{R}^3 \) such that
\[
   W = \text{span}(S).
\]
Deduce, therefore, that \( W \) is a subspace of \( \mathbb{R}^3 \).

**Solution:** Solve the first equation for \( x_1 \) and substitute into the second equation to get that
\[
   \begin{aligned}
   -x_1 + 2x_2 - 3x_3 &= 0 \\
   3x_2 - 2x_3 &= 0.
   \end{aligned}
\]
Next, solve for \( x_2 \) in the second equation in system (3) and substitute into the first equation to get
\[
   \begin{aligned}
   \begin{aligned}
   -x_1 - \frac{2}{3} x_3 &= 0 \\
   3x_2 - 2x_3 &= 0.
   \end{aligned}
   \end{aligned}
\]
Solving for $x_1$ and $x_2$ in system (4) then yields
\[ \begin{cases} x_1 = -\frac{5}{3}x_3 \\ x_2 = \frac{2}{3}x_3. \end{cases} \] (5)

Setting $x_3 = 3t$, where $t$ is an arbitrary parameter, $t$, then gives the solutions
\[ \begin{cases} x_1 = -5t \\ x_2 = 2t \\ x_3 = 3t. \end{cases} \] (6)

We then get that the solution space for the system is
\[ W = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in \mathbb{R}^3 \mid \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = t \begin{pmatrix} -5 \\ 2 \\ 3 \end{pmatrix}, \quad t \in \mathbb{R} \right\}, \]
or
\[ W = \text{span}(S), \]
where
\[ S = \left\{ \begin{pmatrix} -5 \\ 2 \\ 3 \end{pmatrix} \right\}. \]

Since the span of any set is a subspace, it follows that $W$ is a subspace of $\mathbb{R}^3$. \qed