1. Let $W$ denote the solution space of the equation

$$3x_1 + 8x_2 + 2x_3 - x_4 + x_5 = 0$$

Find a linearly independent subset, $S$, of $\mathbb{R}^5$ such that $W = \text{span}(S)$.

**Solution:** Solve for $x_1$ in terms of the other variables to get

$$x_1 = -\frac{8}{3}x_2 - \frac{2}{3}x_3 + \frac{1}{3}x_4 - \frac{1}{3}x_5.$$

Setting $x_2 = -3t$, $x_3 = -3s$, $x_4 = 3r$ and $x_5 = -3q$, where $t, s, r, q$ are arbitrary parameters, we get that

$$x_1 = 8t + 2s + r + q$$
$$x_2 = -3t$$
$$x_3 = -3s$$
$$x_4 = 3r$$
$$x_5 = -3q$$

so that

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = t \begin{pmatrix} 8 \\ -3 \\ 0 \\ 0 \\ 0 \end{pmatrix} + s \begin{pmatrix} 2 \\ 0 \\ -3 \\ 0 \\ 0 \end{pmatrix} + r \begin{pmatrix} 1 \\ 0 \\ 0 \\ 3 \\ 0 \end{pmatrix} + q \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ -3 \end{pmatrix}.$$

Hence, the solution space, $W$, is spanned by the set

$$S = \left\{ \begin{pmatrix} 8 \\ -3 \\ 0 \\ 0 \\ 0 \\ \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ -3 \\ 0 \\ 0 \\ \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ 3 \\ 0 \\ \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ -3 \end{pmatrix} \right\}.$$

To see that $S$ is linearly independent, consider the vector equation

$$c_1 \begin{pmatrix} 8 \\ -3 \\ 0 \\ 0 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 2 \\ 0 \\ -3 \\ 0 \\ 0 \end{pmatrix} + c_3 \begin{pmatrix} 1 \\ 0 \\ 0 \\ 3 \\ 0 \end{pmatrix} + c_4 \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ -3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix},$$
which yields the system

\[
\begin{align*}
8c_1 + 2c_2 + c_3 + c_4 &= 0 \\
-3c_1 &= 0 \\
-3c_2 &= 0 \\
3c_3 &= 0 \\
-3c_4 &= 0.
\end{align*}
\]

From the last four equations we get that

\[c_1 = c_2 = c_3 = c_4 = 0.\]

Hence, \(S\) is linearly independent and \(W = \text{span}(S)\). \(\square\)

2. Let \(W\) denote the solution space of the system

\[
\begin{align*}
x_1 - 2x_2 - x_3 &= 0 \\
2x_1 - 3x_2 + x_3 &= 0.
\end{align*}
\]

Find a linearly independent subset, \(S\), of \(\mathbb{R}^3\) such that \(W = \text{span}(S)\).

**Solution:** Multiplying the first equation by \(-2\), adding the scalar multiple to the second equation, and replacing the second equation by the result yields the system

\[
\begin{align*}
x_1 - 2x_2 - x_3 &= 0 \\
x_2 + 3x_3 &= 0.
\end{align*}
\]  \(\text{(1)}\)

Next, multiply the second equation in system (1) by 2, add the scalar multiple to the first equation and replace the first equation by the result to get

\[
\begin{align*}
x_1 + 5x_3 &= 0 \\
x_2 + 3x_3 &= 0.
\end{align*}
\]  \(\text{(2)}\)

The system in (2) can now be solved for the leading variable \(c_1\) and \(c_2\) to get

\[
\begin{align*}
x_1 &= -5x_3 \\
x_2 &= -3x_3.
\end{align*}
\]  \(\text{(3)}\)

Setting \(x_3 = -t\), where \(t\) is an arbitrary parameter, we obtain the solutions

\[
\begin{align*}
x_1 &= 5t \\
x_2 &= 3t \\
x_3 &= -t.
\end{align*}
\]
It then follows that the solution space of the system is

\[ W = \text{span} \left\{ \begin{pmatrix} 5 \\ 3 \\ -1 \end{pmatrix} \right\}. \]

\[ \square \]

3. In the following system, find the value or values of \( \lambda \) for which the system has nontrivial solutions. In each case, give a linearly independent subset of \( \mathbb{R}^2 \) which generates the solution space.

\[
\begin{aligned}
(\lambda - 3)x + y &= 0 \\
x + (\lambda - 3)y &= 0
\end{aligned}
\]

**Solution:** Solve for \( y \) in the first equation and substitute into the second to get

\[ x - (\lambda - 3)^2x = 0, \]

which factors into

\[ x[1 - (\lambda - 3)^2] = 0. \]

If \( x = 0 \), we get from the first equation that \( y = 0 \), and so we get the trivial solution. Hence, since we are looking for non-trivial solutions, we must have that

\[ 1 - (\lambda - 3)^2 = 0. \]

This quadratic equation can be solved to yield

\[ \lambda - 3 = \pm 1, \]

so that

\[ \lambda = 2 \quad \text{or} \quad \lambda = 4. \]

In the case that \( \lambda = 2 \) we get the system

\[
\begin{aligned}
-x + y &= 0 \\
x - y &= 0
\end{aligned}
\]

which reduces to the equation

\[ x = y. \]
Setting $y = t$, where $t$ is an arbitrary parameter, we get that

\[
\begin{align*}
  x &= t \\
  y &= t
\end{align*}
\]

so that, for $\lambda = 2$, the solution space is

\[
\text{span} \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}.
\]

On the other hand, if $\lambda = 4$, we obtain the system

\[
\begin{align*}
  x + y &= 0 \\
  x + y &= 0
\end{align*}
\]

which reduces to the equation

\[
x = -y.
\]

Setting $y = -t$, where $t$ is an arbitrary parameter, we get that

\[
\begin{align*}
  x &= t \\
  y &= -t
\end{align*}
\]

so that, for $\lambda = 2$, the solution space is

\[
\text{span} \left\{ \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right\}.
\]

\[\square\]

4. Let $v \in \mathbb{R}^n$ and $S$ be a subset of $\mathbb{R}^n$.

(a) Show that the set $\{v\}$ is linearly independent if and only if $v \neq \mathbf{0}$.

Proof: Suppose first that $\{v\}$ is linearly independent. If $v = \mathbf{0}$, then

\[
cv = \mathbf{0}
\]

for any scalar $c$. It then follows that the equation

\[
cv = \mathbf{0}
\]
has nontrivial solutions and therefore \( \{v\} \) is linearly dependent. But this contradicts the assumption of independence. We therefore conclude that \( v \neq 0 \).

Conversely, suppose that \( v \neq 0 \), and consider the equation

\[
.cv = 0.
\]

Since \( v \neq 0 \), we must have that \( c = 0 \) and therefore

\[
.cv = 0
\]

has only the trivial solution \( c = 0 \). Consequently, \( \{v\} \) is linearly independent.

(b) Show that if \( 0 \in S \), then \( S \) is linearly dependent.

Proof: If \( S = \{0\} \), then \( S \) is linearly dependent by part (a). Thus, suppose that \( S \neq \{0\} \). Then, there exists \( v \in S \) such that \( v \neq 0 \). Observe that

\[
.0 = 0 \cdot v
\]

so that \( 0 \) is in the span of \( v \), and therefore \( S \) is linearly dependent.

5. Let \( v_1 \) and \( v_2 \) be vectors in \( \mathbb{R}^n \), and let \( c \) be a scalar.

(a) Show that \( \{v_1, v_2\} \) is linearly independent if and only if \( \{v_1, cv_1 + v_2\} \) is also linearly independent.

Proof: First observe that if \( c = 0 \), \( \{v_1, v_2\} \) and \( \{v_1, cv_1 + v_2\} \) are the same set. So the result holds true in this case. Thus, assume for the rest of the proof that \( c \neq 0 \).

Suppose that that \( \{v_1, v_2\} \) is linearly independent and consider the equation

\[
c_1v_1 + c_2(cv_1 + v_2) = 0.
\]

Using the distributive and associative properties we get that

\[
.(c_1 + cc_2)v_1 + cc_2v_2 = 0.
\]

It then follows from the linear independence of \( \{v_1, v_2\} \) that

\[
c_1 + cc_2 = 0
\]

\[
.cc_2 = 0.
\]
Since $c \neq 0$, we deduce from the above equations that $c_1 = c_2 = 0$ is the only solution of the system. Therefore, the set $\{v_1, cv_1 + v_2\}$ is linearly independent.

Conversely, suppose that $\{v_1, cv_1 + v_2\}$ is linearly independent and consider the vector equation

$$c_1 v_1 + c_2 v_2 = 0.$$  

Adding $cc_2 v_1 - cc_2 v_1 = 0$ to both sides of the equation we get

$$c_1 v_1 + c_2 v_2 + cc_2 v_1 - cc_2 v_1 = 0,$$

which by virtue of the distributive and associative properties can be written as

$$(c_1 - cc_2)v_1 + c_2(cv_1 + v_2) = 0.$$  

Thus, since $\{v_1, cv_1 + v_2\}$ is linearly independent, it follows that

$$c_1 - cc_2 = 0 \quad c_2 = 0,$$

from which we get that $c_1 = c_2 = 0$. Hence, $\{v_1, v_2\}$ is linearly independent. \hfill \Box

(b) Show that

$$\text{span}\{v_1, v_2\} = \text{span}\{v_1, cv_1 + v_2\}.$$  

Proof: Observe that $cv_1 + v_2 \in \text{span}\{v_1, v_2\}$. Therefore,

$$\{v_1, cv_1 + v_2\} \subseteq \text{span}\{v_1, v_2\}.$$  

It then follows that

$$\text{span}\{v_1, cv_1 + v_2\} \subseteq \text{span}\{v_1, v_2\},$$

since $\text{span}\{v_1, cv_1 + v_2\}$ is the smallest subspace of $\mathbb{R}^n$ which contains $\{v_1, cv_1 + v_2\}$. Therefore, it suffices to show that

$$\text{span}\{v_1, v_2\} \subseteq \text{span}\{v_1, cv_1 + v_2\}.$$  

To see why the last inclusion is true, observe that

$$v_2 = v_2 + cv_1 - cv_1 = -cv_1 + (cv_1 + v_2).$$
which is a linear combination of $v_1$ and $cv_1 + v_2$. It then follows that $v_2 \in \text{span}\{v_1, cv_1 + v_2\}$ and therefore

$$\{v_1, v_2\} \subseteq \text{span}\{v_1, cv_1 + v_2\}.$$ 

The last inclusion implies that

$$\text{span}\{v_1, v_2\} \subseteq \text{span}\{v_1, cv_1 + v_2\}$$

because $\text{span}\{v_1, v_2\}$ is the smallest subspace of $\mathbb{R}^n$ which contains $\{v_1, v_2\}$. $\square$