Assignment #7

Due on Wednesday, February 18, 2009

Read Section 2.2 on Gaussian Elimination in Messer (pp. 69–74).
Read Section 2.3 on Solving Linear Systems in Messer (pp. 76–79).
Read Section 3.2 on Span in Messer (pp. 97–102).
Read Section 3.3 on Linear Independence in Messer (pp. 103–109).

Background and Definitions

(Fundamental Theorem of Homogeneous Linear Systems; Theorem 2.6 in Messer, pg. 78). A homogeneous system of $m$ linear equations in $n$ unknowns,

\[
\begin{align*}
    a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= 0 \\
    a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= 0 \\
    \vdots &= \vdots \\
    a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= 0,
\end{align*}
\]

with $n > m$ has at least one nontrivial solution.

Do the following problems

1. Prove that if the homogeneous system in (1) has a nontrivial solution, then it has infinitely many solutions.

2. Consider the vectors $v_1, v_2, v_3$ and $v_4$ in $\mathbb{R}^4$ given by

\[
    v_1 = \begin{pmatrix} 1 \\ 0 \\ -1 \\ 2 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 2 \\ -1 \\ 1 \\ -1 \end{pmatrix}, \quad v_3 = \begin{pmatrix} 0 \\ -1 \\ 3 \\ -5 \end{pmatrix}, \quad \text{and} \quad v_4 = \begin{pmatrix} 1 \\ -3 \\ 0 \\ 1 \end{pmatrix}.
\]

Determine whether the set $\{v_1, v_2, v_3, v_4\}$ is linearly independent; if not, find a linearly independent subset of $\{v_1, v_2, v_3, v_4\}$ which spans $\text{span}\{v_1, v_2, v_3, v_4\}$.

3. Let

\[
    W = \text{span} \left\{ \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 2 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ 4 \\ 3 \end{pmatrix} \right\}.
\]

Find a linearly independent subset of $W$ which spans $W$. 
4. Let $W$ denote the solution space of the system

$$\begin{aligned}
3x_1 - 2x_2 - 2x_3 - x_4 + x_5 &= 0 \\
x_1 - 3x_2 - 2x_5 &= 0 \\
2x_2 + x_3 + 2x_4 - x_5 &= 0 \\
-x_1 + x_2 - x_3 + x_4 - x_5 &= 0.
\end{aligned}$$

Find a linearly independent subset, $S$, of $\mathbb{R}^5$ such that $W = \text{span}(S)$.

5. Determine whether or not the vector $\begin{pmatrix} 4 \\ 7 \\ 7 \\ 4 \end{pmatrix}$ lies in the span of the set $\begin{bmatrix}
\begin{pmatrix} 1 \\ 1 \\ 3 \\ 0 \end{pmatrix},
\begin{pmatrix} 0 \\ -1 \\ 0 \\ 1 \end{pmatrix},
\begin{pmatrix} 1 \\ 2 \\ 3 \\ 3 \end{pmatrix},
\begin{pmatrix} 1 \\ -1 \\ 3 \\ -2 \end{pmatrix}
\end{bmatrix}$. 