Assignment #9

Due on Monday, February 23, 2009

Read Section 3.4 on *Basis* in Messer (pp. 111–113).

Background and Definitions

- *(Definition of basis for a subspace of $\mathbb{R}^n$).* Let W be a subspace of $\mathbb{R}^n$. A subset, B, of W is said to be a **basis** for W if and only if

  (i) B is linearly independent, and

  (ii) $W = \text{span}(B)$.

- *(Column space of a matrix).* The **column space** of a matrix,

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}, \tag{1}$$

  denoted by C_A, is the span of the columns of A. That is,

$$C_A = \text{span}\left\{ \begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{pmatrix}, \begin{pmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{pmatrix}, \ldots, \begin{pmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{pmatrix} \right\}.$$

  Thus, C_A is a subspace of $\mathbb{R}^m$.

- *(Null space of a matrix).* The **null space** of the matrix A defined in (1), denoted by N_A, is the solution space of the homogenous linear system

$$\left\{ \begin{array}{rcl} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n & = & 0 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n & = & 0 \\ \vdots & \vdots & \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n & = & 0. \end{array} \right.$$

  Thus, N_A is a subspace of $\mathbb{R}^n$.

Do the following problems

1. Let
$$W = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 \;\middle|\; 2x + 3y - z = 0 \right\}.$$
Find a basis for W.

2. Let A denote the matrix
$$\begin{pmatrix} 1 & 3 & -1 & 0 \\ 2 & 2 & 2 & 4 \\ 1 & 0 & 2 & 3 \end{pmatrix}. \tag{2}$$
Find a basis for the column space, C_A, of the matrix A.

3. Find a basis for the null space, N_A, of the matrix, A, defined in (2).

4. Given a subset, S, or $\mathbb{R}^n$, and $v \in S$, the expression $S \backslash \{v\}$ denotes the set obtained by removing the vector v from S.

   A subset, S, of a subspace, W, of $\mathbb{R}^n$ is said to be a **minimal generating set** for W iff

   (i) $W = \text{span}(S)$, and

   (ii) for any v in S, the set $S \backslash \{v\}$ does not span W.

   Prove that a minimal generating set for W must be linearly independent.

   Suggestion: Argue by contradiction; that is, start out your argument assuming that S is a minimal generating set for W, but S is linearly dependent. Then, derive a contradiction.

5. Let $\{v_1, v_2, \ldots, v_n\}$ be a subset of n vectors in $\mathbb{R}^n$. Prove that if $\{v_1, v_2, \ldots, v_n\}$ is linearly independent, then it must also span $\mathbb{R}^n$.