Solutions to Assignment #9

1. Let

\[ W = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 \mid 2x + 3y - z = 0 \right\}. \]

Find a basis for \( W \).

**Solution:** \( W \) is the solution space of the homogeneous linear equation

\[ 2x + 3y - z = 0. \]

Solving for \( z \) in terms of \( x \) and \( y \), and setting these to be arbitrary parameters \( t \) and \( s \), respectively, we get the solutions

\[
\begin{align*}
 x &= t \\
 y &= s \\
 z &= 2t + 3s,
\end{align*}
\]

from which we get that

\[ W = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 \mid \begin{pmatrix} x \\ y \\ z \end{pmatrix} = t \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} + s \begin{pmatrix} 0 \\ 1 \\ 3 \end{pmatrix} \right\}. \]

In other words,

\[ W = \text{span} \left\{ \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 3 \end{pmatrix} \right\}. \]

Thus, the set

\[ B = \left\{ \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 3 \end{pmatrix} \right\} \]

is a candidate for a basis for \( W \). To show that \( B \) is a basis, it remains to show that it is linearly independent. So, consider the vector equation

\[
c_1 \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},
\]

which can be solved to find the values of \( c_1 \) and \( c_2 \).
which is equivalent to the system
\[
\begin{align*}
    c_1 & = 0 \\
    c_2 & = 0 \\
    2c_1 + 3c_2 & = 0,
\end{align*}
\]
from which we read that \(c_1 = c_2 = 0\) is the only solution. Consequently, \(B\) is linearly independent.
We therefore conclude that \(B\) is a basis for \(W\). \(\square\)

2. Let \(A\) denote the matrix
\[
\begin{pmatrix}
1 & 3 & -1 & 0 \\
2 & 2 & 2 & 4 \\
1 & 0 & 2 & 3
\end{pmatrix}.
\]

Find a basis for the column space, \(C_A\), of the matrix \(A\).

**Solution:** \(C_A\) is the span of the columns of \(A\):

\[
C_A = \text{span} \left\{ \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 2 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ 4 \\ 3 \end{pmatrix} \right\}.
\]

Denote the columns of \(A\) by \(v_1, v_2, v_3, v_4\), respectively. To find a basis for \(C_A\), we need to find a linearly independent subset of \(\{v_1, v_2, v_3, v_4\}\) which also spans \(C_A\). In order to do this, we seek for nontrivial solutions to the vector equation:
\[
c_1 v_1 + c_2 v_2 + c_3 v_3 + c_4 v_4 = 0,
\]
where 0 denotes the zero–vector in \(\mathbb{R}^3\). This equation is equivalent to the the homogeneous system
\[
\begin{align*}
    c_1 + 3c_2 - c_3 & = 0 \\
    2c_1 + 2c_2 + 2c_3 + 4c_4 & = 0 \\
    c_1 + 2c_3 + 3c_4 & = 0.
\end{align*}
\]

The augmented matrix of this system is:
\[
\begin{pmatrix}
1 & 3 & -1 & 0 & | & 0 \\
2 & 2 & 2 & 4 & | & 0 \\
1 & 0 & 2 & 3 & | & 0
\end{pmatrix}.
\]
We can reduce this matrix to
\[
\begin{pmatrix}
1 & 0 & 2 & 3 & | & 0 \\
0 & 1 & -1 & -1 & | & 0 \\
0 & 0 & 0 & 0 & | & 0 \\
\end{pmatrix},
\]
where we have performed the elementary row operations $\frac{1}{2}R_2 \rightarrow R_2$, $-R_1 + R_2 \rightarrow R_2$, $-R_1 + R_3 \rightarrow R_3$, $-\frac{1}{2}R_2 \rightarrow R_2$, $3R_2 + R_3 \rightarrow R_3$ and $-3R_3 + R_1 \rightarrow R_1$ in succession.

This yields the system
\[
\begin{align*}
    c_1 + 2c_3 + 3c_4 &= 0 \\
    c_2 - c_3 - c_4 &= 0,
\end{align*}
\]
which is equivalent to system (3). Solving for the leading variables in (4) yields the solutions
\[
\begin{align*}
    c_1 &= 2t + 3s \\
    c_2 &= -t - s \\
    c_3 &= -t \\
    c_4 &= -s,
\end{align*}
\]
where $t$ and $s$ are arbitrary parameters. Taking $t = 1$ and $s = 0$ in (5) yields from (2) the linear relation
\[
2v_1 - v_2 - v_3 = 0,
\]
which shows that $v_3 = 2v_1 - v_2$; that is, $v_3 \in \text{span}\{v_1, v_2\}$.

Similarly, taking $t = 0$ and $s = 1$ in (5) yields
\[
3v_1 - v_2 + v_4 = 0,
\]
which shows that $v_4 = -3v_1 + v_2$; that is, $v_4 \in \text{span}\{v_1, v_2\}$.

We then have that both $v_3$ and $v_4$ are in the span of $\{v_1, v_2\}$. Consequently,
\[
\{v_1, v_2, v_3, v_4\} \subseteq \text{span}\{v_1, v_2\},
\]
from which we get that
\[
\text{span}\{v_1, v_2, v_3, v_4\} \subseteq \text{span}\{v_1, v_2\},
\]
since $\text{span}\{v_1, v_2, v_3, v_4\}$ is the smallest subspace of $\mathbb{R}^3$ which contains $\{v_1, v_2, v_3, v_4\}$. Combining this with
\[
\text{span}\{v_1, v_2\} \subseteq \text{span}\{v_1, v_2, v_3, v_4\},
\]
we conclude that
\[ \text{span}\{v_1, v_2\} = \text{span}\{v_1, v_2, v_3, v_4\}; \]
that is \(\{v_1, v_2\}\) spans \(W\). Thus, we set \(B = \{v_1, v_2\}\).

It remains to show that \(B\) is linearly independent. To prove this, consider the vector equation
\[ c_1 v_1 + c_2 v_2 = 0, \quad (6) \]
which leads to the system
\[
\begin{align*}
  c_1 + 2c_2 &= 0 \\
  -c_2 &= 0 \\
  -c_1 + c_2 &= 0 \\
  2c_1 - c_2 &= 0,
\end{align*}
\]
which can be seen to have only the trivial solution: \(c_1 = c_2 = 0\). It then follows that the vector equation (6) has only the trivial solution, and therefore \(B\) is linearly independent. We therefore conclude that the set
\[ B = \left\{ \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ 2 \\ 0 \end{pmatrix} \right\} \]
is a basis for \(C_A\). \(\square\)

3. Find a basis for the null space, \(N_A\), of the matrix, \(A\), defined in (1).

**Solution:** \(N_A\) is the solution space of the homogeneous system
\[
\begin{align*}
  c_1 + 3c_2 - c_3 &= 0 \\
  2c_1 + 2c_2 + 2c_3 + 4c_4 &= 0 \\
  c_1 + 2c_3 + 3c_4 &= 0.
\end{align*}
\]
which is the same as system (3) in the previous problem. Therefore, system (7) is equivalent to the reduced system
\[
\begin{align*}
  c_1 + 2c_3 + 3c_4 &= 0 \\
  c_2 - c_3 - c_4 &= 0.
\end{align*}
\]

Hence, \(N_A\) is the same as the solution space of system (8), which is given by
\[
\begin{align*}
\begin{cases}
  c_1 &= 2t + 3s \\
  c_2 &= -t - s \\
  c_3 &= -t \\
  c_4 &= -s,
\end{cases}
\end{align*}
\]
where \(t\) and \(s\) are arbitrary parameters. Thus,
\[
N_A = \left\{ \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{pmatrix} \in \mathbb{R}^4 \mid \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{pmatrix} = t \begin{pmatrix} 2 \\ -1 \\ -1 \\ 0 \end{pmatrix} + s \begin{pmatrix} 3 \\ -1 \\ 0 \\ -1 \end{pmatrix} \right\},
\]
or
\[
N_A = \text{span} \left\{ \begin{pmatrix} 2 \\ -1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 \\ -1 \\ 0 \\ -1 \end{pmatrix} \right\}.
\]
Set
\[
B = \left\{ \begin{pmatrix} 2 \\ -1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 \\ -1 \\ 0 \\ -1 \end{pmatrix} \right\}.
\]
Then, \(B\) spans \(N_A\) and is also linearly independent. Therefore, \(B\) is a basis for \(N_A\). \(\square\)

4. Given a subset, \(S\), or \(\mathbb{R}^n\), and \(v \in S\), the expression \(S\backslash\{v\}\) denotes the set obtained by removing the vector \(v\) from \(S\).

A subset, \(S\), of a subspace, \(W\), of \(\mathbb{R}^n\) is said to be a minimal generating set for \(W\) iff
\begin{itemize}
  \item[(i)] \(W = \text{span}(S)\), and
  \item[(ii)] for any \(v\) in \(S\), the set \(S\backslash\{v\}\) does not span \(W\).
\end{itemize}

Prove that a minimal generating set for \(W\) must be linearly independent.

\textit{Suggestion:} Argue by contradiction; that is, start out your argument assuming that \(S\) is a minimal generating set for \(W\), but \(S\) is linearly dependent. Then, derive a contradiction.
Proof: Assume that $S$ is a subset of $W$ which satisfies (i) and (ii) above. Suppose by way of contradiction that $S$ is not linearly independent. Then, one of the vectors in $S$, call it $v$, is in the span of the other ones; that is,

$$v \in \text{span}(S\{v\}).$$

It then follows that

$$S \subseteq \text{span}(S\{v\}),$$

from which we get that

$$\text{span}(S) \subseteq \text{span}(S\{v\}),$$

(9)

since $\text{span}(S)$ is the smallest subspace of $\mathbb{R}^n$ which contains $S$. On the other hand, since $S\{v\} \subseteq S$, we also get that

$$\text{span}(S\{v\}) \subseteq \text{span}(S).$$

Combining this with (9) we get that

$$\text{span}(S\{v\}) = \text{span}(S).$$

Thus, since $S$ satisfies (i),

$$\text{span}(S\{v\}) = W.$$ 

But this contradicts (ii). We therefore conclude that $S$ is linearly independent, which was to be shown. \qed

5. Let $\{v_1, v_2, \ldots, v_n\}$ be a subset of $n$ vectors in $\mathbb{R}^n$. Prove that if $\{v_1, v_2, \ldots, v_n\}$ is linearly independent, then it must also span $\mathbb{R}^n$.

Proof: Assume that $\{v_1, v_2, \ldots, v_n\}$ is linearly independent. Arguing by contradiction, suppose that $\{v_1, v_2, \ldots, v_n\}$ does not span $\mathbb{R}^n$. Then, there exists $v \in \mathbb{R}^n$ such that

$$v \not\in \text{span}\{v_1, v_2, \ldots, v_n\}.$$ 

Consequently, the set $\{v_1, v_2, \ldots, v_n, v\}$ is linearly independent. However, $\{v_1, v_2, \ldots, v_n, v\}$ contains $n + 1$ vectors; therefore, it must be linearly dependent. We have therefore arrived at a contradiction. Hence, $\{v_1, v_2, \ldots, v_n\}$ must also span $\mathbb{R}^n$. \qed