Solutions to Review Problems for Exam 1

1. Consider the set $B = \left\{ \begin{pmatrix} 1 \\ 1 \\ \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ \end{pmatrix} \right\}$.

(a) Show that $B$ is a basis for $\mathbb{R}^2$.

Proof: Given that $\text{dim}(\mathbb{R}^2) = 2$ and that $B$ contains two vectors, to prove that $B$ is a basis for $\mathbb{R}^2$, it suffices to prove that $B$ is linearly independent. Thus, consider the vector equation

$$c_1 \begin{pmatrix} 1 \\ 1 \\ \end{pmatrix} + c_2 \begin{pmatrix} -1 \\ 1 \\ \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \end{pmatrix},$$

which is equivalent to the system

$$\begin{cases} c_1 - c_2 = 0 \\ c_1 + c_2 = 0. \end{cases}$$

The system in (2) can be solved to yield the unique solution $c_1 = c_2 = 0$. Hence, the vector equation in (1) has only the trivial solution, and therefore $B$ is linearly independent.

(b) Give the coordinates of the vector $v = \begin{pmatrix} 1 \\ 0 \\ \end{pmatrix}$ relative to $B$. Interpret your result geometrically.

Solution: We look for scalars, $c_1$ and $c_2$, such that

$$c_1 \begin{pmatrix} 1 \\ 1 \\ \end{pmatrix} + c_2 \begin{pmatrix} -1 \\ 1 \\ \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ \end{pmatrix},$$

This is equivalent to solving the system

$$\begin{cases} c_1 - c_2 = 1 \\ c_1 + c_2 = 0. \end{cases}$$

To solve this system, we may reduce the corresponding augmented matrix,

$$\begin{pmatrix} 1 & -1 & | & 1 \\ 1 & 1 & | & 0 \end{pmatrix},$$

to

$$\begin{pmatrix} 1 & 0 & | & 1/2 \\ 0 & 1 & | & -1/2 \end{pmatrix}.$$
We therefore get that the coordinate vector of \( \begin{pmatrix} 1 \\ 0 \end{pmatrix} \) relative to \( B \) is
\[
\begin{pmatrix} 1 \\ 0 \end{pmatrix} 
\begin{pmatrix} 1/2 \\ -1/2 \end{pmatrix}.
\]
Denote the vectors in \( B \) by \( v_1 \) and \( v_2 \), respectively and in that order, and denote the vector \( \begin{pmatrix} 1 \\ 0 \end{pmatrix} \) by \( v \). Figure 1 shows the vector \( v \) as the sum of the vectors \( \frac{1}{2}v_1 \) and \( -\frac{1}{2}v_2 \).

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{coordinates.png}
\caption{Coordinates relative to \( B \)}
\end{figure}

2. Give a basis for the span of the following set of vectors in \( \mathbb{R}^4 \)

\[
\left\{ \begin{pmatrix} 1 \\ -1 \\ 1 \\ 1 \end{pmatrix} , \begin{pmatrix} -2 \\ 0 \\ 3 \\ 0 \end{pmatrix} , \begin{pmatrix} 1 \\ -3 \\ 6 \\ -3 \end{pmatrix} , \begin{pmatrix} 1 \\ 1 \\ -4 \\ 1 \end{pmatrix} \right\}.
\]

**Solution:** Denote the vectors in the set

\[
\left\{ \begin{pmatrix} 1 \\ -1 \\ 1 \\ 1 \end{pmatrix} , \begin{pmatrix} -2 \\ 0 \\ 3 \\ 0 \end{pmatrix} , \begin{pmatrix} 1 \\ -3 \\ 6 \\ -3 \end{pmatrix} , \begin{pmatrix} 1 \\ 1 \\ -4 \\ 1 \end{pmatrix} \right\}
\]
by \(v_1, v_2, v_3\) and \(v_4\), respectively, we look for a linear vector relation of the form
\[
c_1v_1 + c_2v_2 + c_3v_3 + c_4v_4 = 0. \tag{4}
\]
This leads to the system
\[
\begin{cases}
c_1 - 2c_2 + c_3 + c_4 &= 0 \\
c_1 - 3c_3 + c_4 &= 0 \\
c_1 + 3c_2 + 6c_3 - 4c_4 &= 0 \\
-c_1 - 3c_3 + c_4 &= 0.
\end{cases} \tag{5}
\]
The augmented matrix of this system is:
\[
\begin{bmatrix}
R_1 & 1 & -2 & 1 & 1 & | & 0 \\
R_2 & -1 & 0 & -3 & 1 & | & 0 \\
R_3 & 1 & 3 & 6 & -4 & | & 0 \\
R_4 & -1 & 0 & -3 & 1 & | & 0
\end{bmatrix}
\]
We can reduce this matrix to
\[
\begin{bmatrix}
1 & 0 & 3 & -1 & | & 0 \\
0 & 1 & 1 & -1 & | & 0 \\
0 & 0 & 0 & 0 & | & 0 \\
0 & 0 & 0 & 0 & | & 0
\end{bmatrix}
\]
which is in reduced row–echelon form. We therefore get that the system in (5) is equivalent to the system
\[
\begin{cases}
c_1 + 3c_3 - c_4 &= 0 \\
c_2 + c_3 - c_4 &= 0.
\end{cases} \tag{6}
\]
Solving for the leading variables in (6) yields the solutions
\[
\begin{cases}
c_1 &= 3t + s \\
c_2 &= t + s \\
c_3 &= -t \\
c_4 &= s,
\end{cases} \tag{7}
\]
where \(t\) and \(s\) are arbitrary parameters. Taking \(t = 1\) and \(s = 0\) in (7) yields from (4) the linear relation
\[
3v_1 + v_2 - v_3 = 0,
\]
which shows that \(v_3 = -3v_1 - v_2\); that is, \(v_3 \in \text{span}\{v_1, v_2\}\).
Similarly, taking $t = 0$ and $s = 1$ in (7) yields

$$v_1 + v_2 + v_4 = 0,$$

which shows that $v_4 = -v_1 - v_2$; that is, $v_4 \in \text{span}\{v_1, v_2\}$.

We then have that both $v_3$ and $v_4$ are in the span of $\{v_1, v_2\}$. Consequently,

$$\{v_1, v_2, v_3, v_4\} \subseteq \text{span}\{v_1, v_2\},$$

from which we get that

$$\text{span}\{v_1, v_2, v_3, v_4\} \subseteq \text{span}\{v_1, v_2\},$$

since $\text{span}\{v_1, v_2, v_3, v_4\}$ is the smallest subspace of $\mathbb{R}^3$ which contains $\{v_1, v_2, v_3, v_4\}$. Combining this with

$$\text{span}\{v_1, v_2\} \subseteq \text{span}\{v_1, v_2, v_3, v_4\},$$

we conclude that

$$\text{span}\{v_1, v_2\} = \text{span}\{v_1, v_2, v_3, v_4\};$$

that is, $\{v_1, v_2\}$ spans $\text{span}\{v_1, v_2, v_3, v_4\}$.

To see that $\{v_1, v_2\}$ is linearly independent, observe that $v_1$ and $v_2$ are not multiples of each other. We therefore conclude that $\{v_1, v_2\}$ is a basis for $\text{span}\{v_1, v_2, v_3, v_4\}$. $\square$

3. Find a basis for the solution space of the system

\[
\begin{cases}
  x_1 - x_2 + x_3 - x_4 = 0 \\
  2x_1 - x_2 - 2x_4 = 0 \\
  -x_1 + x_3 + x_4 = 0,
\end{cases}
\]

and compute its dimension.

**Solution:** We first find the solution space, $W$, of the system. In order to do this, we reduce the augmented matrix of this system,

\[
\begin{pmatrix}
  R_1 & 1 & -1 & 1 & -1 & | & 0 \\
  R_2 & 2 & -1 & 0 & -2 & | & 0 \\
  R_3 & -1 & 0 & 1 & 1 & | & 0
\end{pmatrix},
\]

Similarly, taking $t = 0$ and $s = 1$ in (7) yields

$$v_1 + v_2 + v_4 = 0,$$

which shows that $v_4 = -v_1 - v_2$; that is, $v_4 \in \text{span}\{v_1, v_2\}$.

We then have that both $v_3$ and $v_4$ are in the span of $\{v_1, v_2\}$. Consequently,

$$\{v_1, v_2, v_3, v_4\} \subseteq \text{span}\{v_1, v_2\},$$

from which we get that

$$\text{span}\{v_1, v_2, v_3, v_4\} \subseteq \text{span}\{v_1, v_2\},$$

since $\text{span}\{v_1, v_2, v_3, v_4\}$ is the smallest subspace of $\mathbb{R}^3$ which contains $\{v_1, v_2, v_3, v_4\}$. Combining this with

$$\text{span}\{v_1, v_2\} \subseteq \text{span}\{v_1, v_2, v_3, v_4\},$$

we conclude that

$$\text{span}\{v_1, v_2\} = \text{span}\{v_1, v_2, v_3, v_4\};$$

that is, $\{v_1, v_2\}$ spans $\text{span}\{v_1, v_2, v_3, v_4\}$.

To see that $\{v_1, v_2\}$ is linearly independent, observe that $v_1$ and $v_2$ are not multiples of each other. We therefore conclude that $\{v_1, v_2\}$ is a basis for $\text{span}\{v_1, v_2, v_3, v_4\}$. $\square$

3. Find a basis for the solution space of the system

\[
\begin{cases}
  x_1 - x_2 + x_3 - x_4 = 0 \\
  2x_1 - x_2 - 2x_4 = 0 \\
  -x_1 + x_3 + x_4 = 0,
\end{cases}
\]

and compute its dimension.

**Solution:** We first find the solution space, $W$, of the system. In order to do this, we reduce the augmented matrix of this system,
to its reduced row–echelon form:

\[
\begin{pmatrix}
1 & 0 & -1 & 0 & | & 0 \\
0 & 1 & -2 & 0 & | & 0 \\
0 & 0 & 0 & 1 & | & 0
\end{pmatrix}.
\]

Consequently, the system in (8) is equivalent to the system

\[
\begin{cases}
x_1 - x_3 = 0 \\
x_2 - 2x_3 = 0 \\
x_4 = 0.
\end{cases}
\]

Solving for the leading variables in the system in (9) we obtain the solutions

\[
\begin{cases}
x_1 = t \\
x_2 = 2t \\
x_3 = t \\
x_4 = 0,
\end{cases}
\]

where \( t \) is an arbitrary parameter. I then follows that the solution space of system (9) is

\[W = \text{span} \left\{ \begin{pmatrix} 1 \\ 2 \\ 1 \\ 0 \end{pmatrix} \right\}.\]

Hence

\[
\begin{cases}
\begin{pmatrix} 1 \\ 2 \\ 1 \\ 0 \end{pmatrix}
\end{cases}
\]

is a basis for \( W \) and therefore \( \dim(W) = 1. \)

4. Prove that any set of four vectors in \( \mathbb{R}^3 \) must be linearly dependent.

\textbf{Proof:} Let \( v_1, v_2, v_3 \) and \( v_4 \) denote four vectors in \( \mathbb{R}^3 \) and write

\[
v_1 = \begin{pmatrix} a_{11} \\ a_{21} \\ a_{31} \end{pmatrix}, \quad v_2 = \begin{pmatrix} a_{12} \\ a_{22} \\ a_{32} \end{pmatrix}, \quad v_3 = \begin{pmatrix} a_{13} \\ a_{23} \\ a_{33} \end{pmatrix} \quad \text{and} \quad v_4 = \begin{pmatrix} a_{14} \\ a_{24} \\ a_{34} \end{pmatrix}.
\]

Consider the vector equation

\[
c_1v_1 + c_2v_2 + c_3v_3 + c_4v_4, = 0.
\]

(10)
This equation translates into the homogeneous system
\[
\begin{align*}
  a_{11}c_1 + a_{12}c_2 + a_{13}c_3 + a_{14}c_4 &= 0 \\
  a_{21}c_1 + a_{22}c_2 + a_{23}c_3 + a_{24}c_4 &= 0 \\
  a_{31}c_1 + a_{32}c_2 + a_{33}c_3 + a_{34}c_4 &= 0,
\end{align*}
\]

(11)
of 3 linear equations in 4 unknowns. It then follows from the Fundamental Theorem for homogeneous linear systems that system (11) has infinitely many solutions. Consequently, the vector equation in (10) has a nontrivial solution, and therefore the set \(\{v_1, v_2, v_3, v_4\}\) is linearly dependent.

5. Show that if the set \(\{v_1, v_2\}\) is a linearly independent subset of \(\mathbb{R}^n\), then so is the set \(\{v_1, cv_1 + v_2\}\), where \(c\) is a scalar, and, conversely, if \(\{v_1, cv_1 + v_2\}\) is linearly independent, then so is \(\{v_1, v_2\}\). Show also that \(\text{span}\{v_1, v_2\} = \text{span}\{v_1, cv_1 + v_2\}\).

(a) First we prove that \(\{v_1, v_2\}\) is a linearly independent subset of \(\mathbb{R}^n\), then so is the set \(\{v_1, cv_1 + v_2\}\).

Proof: Assume that \(\{v_1, v_2\}\) is a linearly independent and consider the vector equation
\[
c_1v_1 + c_2(cv_1 + v_2) = 0.
\]
(12)
Applying the distributive and associative properties, the equation in (12) turns into
\[
(c_1 + cc_2)v_1 + c_2v_2 = 0.
\]
(13)
It follows from (13) and the linear independence of \(\{v_1, v_2\}\) that
\[
\begin{align*}
  c_1 + cc_2 &= 0 \\
  c_2 &= 0.
\end{align*}
\]
(14)
The system in (14) has only the trivial solution: \(c_2 = c_1 = 0\). Hence, the vector equation in (12) has only the trivial solution and therefore the set \(\{v_1, cv_1 + v_2\}\) is linearly independent.

(b) Next, we prove the converse of the statement in (a): If \(\{v_1, cv_1 + v_2\}\) is linearly independent, then \(\{v_1, v_2\}\) is a linearly independent.

Proof: Assume that \(\{v_1, cv_1 + v_2\}\) is a linearly independent and consider the vector equation
\[
c_1v_1 + c_2v_2 = 0.
\]
(15)
Adding \( 0 = cc_2 v_1 - cc_2 v_1 \) to the left-hand side of the equation in (15) and applying the distributive and associative properties we get

\[(c_1 - cc_2)v_1 + c_2(cv_1 + v_2) = 0.\] (16)

It follows from (16) and the linear independence of \( \{v_1, cv_1 + v_2\} \) that

\[
\begin{aligned}
&c_1 - cc_2 = 0 \\
c_2 &\quad = 0.
\end{aligned}
\] (17)

The system in (17) has only the trivial solution: \( c_2 = c_1 = 0 \). Hence, the vector equation in (15) has only the trivial solution and therefore the set \( \{v_1, v_2\} \) is linearly independent.

(c) We prove that that \( \text{span}\{v_1, v_2\} = \text{span}\{v_1, cv_1 + v_2\} \).

\textbf{Proof:} Let \( W = \text{span}\{v_1, v_2\} \). Then, \( W \) is a subspace which contains \( v_1 \) and \( v_2 \) and all their linear combinations; in particular, \( cv_1 + v_2 \in W \). We then have that

\( \{v_1, cv_1 + v_2\} \subseteq W \).

It then follows that

\( \text{span}\{v_1, cv_1 + v_2\} \subseteq W \),

since \( \text{span}\{v_1, cv_1 + v_2\} \) is the smallest subspace of \( \mathbb{R}^n \) which contains \( \{v_1, cv_1 + v_2\} \). On the other hand, for any \( u \in W \) there exist scalars \( c_1 \) and \( c_2 \) such that

\( u = c_1 v_1 + c_2 v_2 \).

Consequently,

\[ u = c_1 v_1 + c_2 v_2 + cc_2 v_1 - cc_2 v_1 = (c_1 - cc_2)v_1 + c_2(cv_1 + v_2), \]

which shows that \( u \in \text{span}\{v_1, cv_1 + v_2\} \); thus,

\( u \in W \Rightarrow u \in \text{span}\{v_1, cv_1 + v_2\} \),

or

\( W \subseteq \text{span}\{v_1, cv_1 + v_2\} \).

Combining this with (18) yields that

\( W = \text{span}\{v_1, cv_1 + v_2\} \).

\[\square\]
6. Let \( J \) and \( H \) be planes in \( \mathbb{R}^3 \) given by
\[
J = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \mid 2x + 3y - 6z = 0 \right\} \quad \text{and} \quad H = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \mid x - 2y + z = 0 \right\}.
\]

(a) Give bases for \( J \) and \( H \) and compute their dimensions.

**Solution:** To find a basis for \( J \), we solve the equation
\[
2x + 3y + z = 0
\]
to get the solution space \( J = \text{span} \left\{ \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix} \right\} \). Thus, the set
\[
\left\{ \begin{pmatrix} 3 \\ 0 \\ 1 \\ \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix} \right\}
\]
is a basis for \( J \) and so \( \dim(J) = 2 \).

Similarly, for \( H \), we solve
\[
x - 2y + z = 0
\]
and obtain that
\[
\left\{ \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \right\}
\]
is a basis for \( H \); thus, \( \dim(H) = 2 \). \( \square \)

(b) Give a basis for the subspace \( J \cap H \) and compute \( \dim(J \cap H) \).

**Solution:** Vectors \( \begin{pmatrix} x \\ y \\ z \end{pmatrix} \) in the intersection of \( J \) and \( H \) if they are solutions to the system of equations
\[
\begin{aligned}
2x + 3y - 6z &= 0 \\
x - 2y + z &= 0.
\end{aligned}
\]

Thus, to find \( J \cap H \), we may elementary row operations on the augmented matrix
\[
\begin{pmatrix}
R_1 & 2 & 3 & -6 & 0 \\
R_2 & 1 & -2 & 1 & 0
\end{pmatrix}
\]
to obtain the reduced matrix
\[
\begin{pmatrix}
1 & 0 & -9/7 & 0 \\
0 & 1 & -8/7 & 0
\end{pmatrix}.
\]
Thus, the system in (19) is equivalent to
\[
\begin{aligned}
x - \frac{9}{7}z &= 0 \\
y - \frac{8}{7}z &= 0,
\end{aligned}
\]
Solving for the leading variables in system (20) and setting \(z = 7t\), where \(t\) is an arbitrary parameter, we obtain that
\[
J \cap H = \text{span} \begin{pmatrix} 9 \\ 8 \\ 7 \end{pmatrix}.
\]
Thus, the set
\[
\left\{ \begin{pmatrix} 9 \\ 8 \\ 7 \end{pmatrix} \right\}
\]
is a basis for \(J \cap H\) and, therefore, \(\dim(J \cap H) = 1\).

7. Let \(W\) be a subspace of \(\mathbb{R}^n\).

(a) Prove that if \(v \in W\) and \(v \neq 0\), then \(rv = sv\) implies that \(r = s\), where \(r\) and \(s\) are scalars.

Proof: Suppose that \(v \in W\), where \(W\) is a subspace of \(\mathbb{R}^n\), and that \(v \neq 0\). Suppose also that
\[
rv = sv
\]
for some scalars \(r\) and \(s\). Add \(-sv\) on both sides of the vector equation in (21) and apply the distributive property to obtain
\[
(r - s)v = 0.
\]
Taking the Euclidean inner product with \(v\) of both sides of (22) yields
\[
(r - s)(v, v) = 0, \quad \text{(23)}
\]
where we have used the bi-linearity of the inner product. It then follows from (23), the positive definiteness of the inner product, and the assumption that \(v \neq 0\), that
\[
r - s = 0
\]
and therefore \(r = s\), which was to be shown.
(b) Prove that if \( W \) has more than one element, then \( W \) has infinitely many elements.

\textit{Proof:} Since \( W \) has at least two elements, there has to be a vector, \( v \), in \( W \) such that \( v \neq 0 \). Now, for any \( t \in \mathbb{R} \), \( tv \in W \) because \( W \) is closed under scalar multiplication. By part (a), \( t_1v \neq t_2v \) for any \( t_1 \neq t_2 \). Consequently, \( W \) contains infinitely many vectors. \( \square \)

8. Let \( W \) be a subspace of \( \mathbb{R}^n \) and \( S_1 \) and \( S_2 \) be subsets of \( W \).

(a) Show that \( \text{span}(S_1 \cap S_2) \subseteq \text{span}(S_1) \cap \text{span}(S_2) \).

\textit{Proof:} First observe that \( S_1 \cap S_2 \subseteq S_1 \) and \( S_1 \cap S_2 \subseteq S_2 \). Consequently, \( \text{span}(S_1 \cap S_2) \subseteq \text{span}(S_1) \) and \( \text{span}(S_1 \cap S_2) \subseteq \text{span}(S_2) \).

It then follows that
\[
\text{span}(S_1 \cap S_2) \subseteq \text{span}(S_1) \cap \text{span}(S_2),
\]
which was to be shown. \( \square \)

(b) Give an example in which \( \text{span}(S_1 \cap S_2) \neq \text{span}(S_1) \cap \text{span}(S_2) \).

\textit{Solution:} Let \( S_1 = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\} \) and \( S_2 = \left\{ \begin{pmatrix} -1 \\ 0 \end{pmatrix} \right\} \). Then, \( S_1 \cap S_2 = \emptyset \) so that \( \text{span}(S_1 \cap S_2) = \{0\} \), where \( 0 \) denotes the zero vector in \( \mathbb{R}^2 \).

On the other hand,
\[
\text{span}(S_1) = \left\{ \begin{pmatrix} x \\ 0 \end{pmatrix} \in \mathbb{R}^2 \mid x \in \mathbb{R} \right\}
\]
and
\[
\text{span}(S_2) = \left\{ \begin{pmatrix} x \\ 0 \end{pmatrix} \in \mathbb{R}^2 \mid x \in \mathbb{R} \right\}.
\]

Hence,
\[
\text{span}(S_1) \cap \text{span}(S_2) = \left\{ \begin{pmatrix} x \\ 0 \end{pmatrix} \in \mathbb{R}^2 \mid x \in \mathbb{R} \right\} \neq \{0\}.
\]
\( \square \)
(c) Show that if \( S_1 \subseteq S_2 \) and \( S_2 \) is linearly independent, then \( S_1 \) is also linearly independent.

Proof: Suppose that \( S_1 \subseteq S_2 \) and \( S_2 \) is linearly independent, and that \( c_1, c_2, \ldots, c_n \) solve the vector equation

\[
c_1v_1 + c_2v_2 + \cdots + c_kv_k = 0,
\]

where \( v_1, v_2, \ldots, v_k \) are vectors in \( S_1 \). Since we are assuming that \( S_1 \subseteq S_2 \), the vectors \( v_1, v_2, \ldots, v_k \) are also in \( S_2 \), which is assumed to be linearly independent; consequently,

\[
c_1 = c_2 = \cdots = c_k = 0.
\]

Thus, we have shown that for any finite set of vectors, \( v_1, v_2, \ldots, v_k \), in \( S_1 \), the vector equation in (24) has only the trivial solution. Hence, \( S_1 \) is linearly independent.

(d) Show that if \( S_1 \subseteq S_2 \) and \( S_1 \) is linearly dependent, then \( S_2 \) is also linearly dependent.

Proof: Suppose that \( S_1 \subseteq S_2 \) and \( S_1 \) is linearly dependent. Then, there exist vectors \( v_1, v_2, \ldots, v_k \) are vectors in \( S_1 \) such that the equation

\[
c_1v_1 + c_2v_2 + \cdots + c_kv_k = 0,
\]

has a nontrivial solution. Since we are assuming that \( S_1 \subseteq S_2 \), the vectors \( v_1, v_2, \ldots, v_k \) are also in \( S_2 \). This proves that \( S_2 \) is linearly dependent.

9. Let \( W_1 \) and \( W_2 \) be two subspaces of \( \mathbb{R}^n \). We write \( W_1 \oplus W_2 \) for the subspace \( W_1 + W_2 \) for the special case in which \( V = W_1 \cap W_2 = \{0\} \). Show that every vector \( v \in W_1 \oplus W_2 \) can be written in the form \( v = v_1 + v_2 \), where \( v_1 \in W_1 \) and \( v_2 \in W_2 \), in one and only one way; that is, if \( v = u_1 + u_2 \), where \( u_1 \in W_1 \) and \( u_2 \in W_2 \), then \( u_1 = v_1 \) and \( u_2 = v_2 \).

Proof: Suppose that \( W_1 \) and \( W_2 \) are two subspaces of \( \mathbb{R}^n \) which have only the zero vector in common; that is, \( W_1 \cap W_2 = \{0\} \). Let \( v \) be any \( v \in W_1 + W_2 \). Then, \( v = v_1 + v_2 \), where \( v_1 \in W_1 \) and \( v_2 \in W_2 \). Suppose that \( v \) can also be written as \( v = u_1 + u_2 \), where \( u_1 \in W_1 \) and \( u_2 \in W_2 \). Then,

\[
v_1 + v_2 = u_1 + u_2,
\]
from which we get that
\[ v_1 - u_1 = v_2 - u_2, \]
(26)
where \( v_1 - u_1 \in W_1 \) and \( v_2 - u_2 \in W_2 \) since \( W_1 \) and \( W_2 \) are subspaces of \( \mathbb{R}^n \).
It also follows from (26) that \( v_1 - u_1 \in W_2 \). Thus, \( v_1 - u_1 \in W_1 \cap W_2 = \{0\} \),
which implies that
\[ v_1 - u_1 = 0, \]
or
\[ v_1 = u_1. \]
Similarly, we get that \( v_2 = u_2 \).

10. Let \( v \in \mathbb{R}^n \) and define \( W = \{ w \in \mathbb{R}^n \mid \langle w, v \rangle = 0 \} \).

(a) Prove that \( W \) is a subspace of \( \mathbb{R}^n \).

Proof: First, observe that \( W \neq \emptyset \) because \( \langle 0, v \rangle = 0 \) and therefore \( 0 \in W \) and so \( W \) is nonempty.
Next, we show that \( W \) is closed under addition and scalar multiplication.
To see that \( W \) is closed under scalar multiplication, observe that, by the bi–linearity property of the inner product, if \( w \in W \), then
\[ \langle t v, w \rangle = t \langle v, w \rangle = t \cdot 0 = 0 \]
for all \( t \in \mathbb{R} \).
To show that \( W \) is closed under vector addition, let \( w_1 \) and \( w_2 \) be two vectors in \( W \). Then, applying the bi–linearity property of the inner product again,
\[ \langle w_1 + w_2, v \rangle = \langle w_1, v \rangle + \langle w_2, v \rangle = 0 + 0 = 0; \]
hence, \( w_1 + w_2 \in W \).

(b) Suppose that \( v \neq 0 \) and compute \( \dim(W) \).

Solution: Let \( B = \{ w_1, w_2, \ldots, w_k \} \) be a basis for \( W \). Then, \( \dim(W) = k \) and we would like to determine what \( k \) is.
First note that \( v \not\in \text{span}(B) \). For, suppose that \( v \in \text{span}(B) = W \), then
\[ \langle v, v \rangle = 0. \]
Thus, by the positive definiteness of the Euclidean inner product, it follows that \( v = 0 \), but we are assuming that \( v \neq 0 \). Consequently, the set
\[ B \cup \{v\} = \{ w_1, w_2, \ldots, w_k, v \} \]
is linearly independent. We claim that $B \cup \{v\}$ also spans $\mathbb{R}^n$. To see why this is so, let $u \in \mathbb{R}^n$ be any vector in $\mathbb{R}^n$, and let

$$t = \frac{\langle u, v \rangle}{\|v\|^2}.$$ 

Write

$$u = tv + (u - tv),$$

and observe that $u - tv \in W$. To see why this is so, compute

$$\langle u - tv, v \rangle = \langle u, v \rangle - t\langle v, v \rangle = \langle u, v \rangle - t\|v\|^2 = \langle u, v \rangle - \frac{\langle u, v \rangle}{\|v\|^2}\|v\|^2 = \langle u, v \rangle - \langle u, v \rangle = 0.$$

Thus, $u - tv \in W$. It then follows that there exist scalars $c_1, c_2, \ldots, c_k$ such that

$$u - tv = c_1 w_1 + c_2 w_2 + \cdots + c_k w_k.$$

Thus,

$$u = c_1 w_1 + c_2 w_2 + \cdots + c_k w_k + tv,$$

which shows that $u \in \text{span}(B \cup \{v\})$. Consequently, $B \cup \{v\}$ spans $\mathbb{R}^n$. Therefore, since $B \cup \{v\}$ is also linearly independent, it forms a basis for $\mathbb{R}^n$. We then have that $B \cup \{v\}$ must have $n$ vectors in it, since $\dim(\mathbb{R}^n) = n$; that is,

$$k + 1 = n,$$

from which we get that

$$\dim(W) = n - 1.$$