Solutions to Review Problems for Exam 1

1. Consider the set $B = \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \end{pmatrix} \right\}$.

   (a) Show that B is a basis for $\mathbb{R}^2$.

   Proof: Given that $\dim(\mathbb{R}^2) = 2$ and that B contains two vectors, to prove that B is a basis for $\mathbb{R}^2$, it suffices to prove that B is linearly independent. Thus, consider the vector equation

   $$c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \tag{1}$$

   which is equivalent to the system

   $$\left\{ \begin{array}{rcl} c_1 - c_2 & = & 0 \\ c_1 + c_2 & = & 0. \end{array} \right. \tag{2}$$

   The system in (2) can be solved to yield the unique solution $c_1 = c_2 = 0$. Hence, the vector equation in (1) has only the trivial solution, and therefore B is linearly independent. $\square$

   (b) Give the coordinates of the vector $v = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ relative to B. Interpret your result geometrically.

   Solution: We look for scalars, c_1 and c_2, such that

   $$c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \tag{3}$$

   This is equivalent to solving the system

   $$\left\{ \begin{array}{rcl} c_1 - c_2 & = & 1 \\ c_1 + c_2 & = & 0. \end{array} \right.$$

   To solve this system, we may reduce the corresponding augmented matrix,

   $$\left(\begin{array}{rr|r} 1 & -1 & 1 \\ 1 & 1 & 0 \end{array} \right),$$

   to

   $$\left(\begin{array}{rr|r} 1 & 0 & 1/2 \\ 0 & 1 & -1/2 \end{array} \right).$$

We therefore get that the coordinate vector of $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ relative to B is

$$\left[\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right]_B = \begin{pmatrix} 1/2 \\ -1/2 \end{pmatrix}.$$

Denote the vectors in B by v_1 and v_2, respectively and in that order, and denote the vector $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ by v. Figure 1 shows the vector v as the sum of the vectors $\frac{1}{2}v_1$ and $-\frac{1}{2}v_2$.

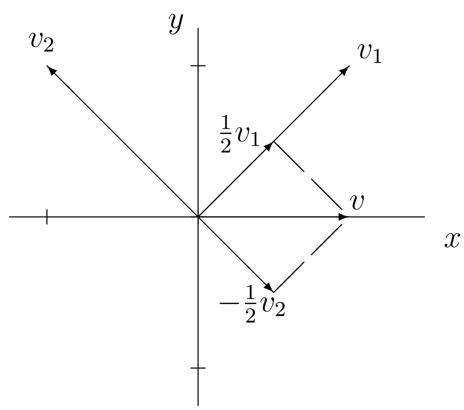

Figure 1: Coordinates relative to B

□

2. Give a basis for the span of the following set of vectors in $\mathbb{R}^4$

$$\left\{ \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} -2 \\ 0 \\ 3 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ -3 \\ 6 \\ -3 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ -4 \\ 1 \end{pmatrix} \right\}.$$

***Solution**:* Denote the vectors in the set

$$\left\{ \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} -2 \\ 0 \\ 3 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ -3 \\ 6 \\ -3 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ -4 \\ 1 \end{pmatrix} \right\}$$

by v_1, v_2, v_3 and v_4, respectively, we look for a linear vector relation of the form

$$c_1 v_1 + c_2 v_2 + c_3 v_3 + c_4 v_4 = \mathbf{0}. \tag{4}$$

This leads to the system

$$\left\{\begin{array}{rcl} c_1 - 2c_2 + c_3 + c_4 & = & 0 \\ -c_1 - 3c_3 + c_4 & = & 0 \\ c_1 + 3c_2 + 6c_3 - 4c_4 & = & 0 \\ -c_1 - 3c_3 + c_4 & = & 0. \end{array}\right. \tag{5}$$

The augmented matrix of this system is:

$$\begin{array}{c} R_1 \\ R_2 \\ R_3 \\ R_4 \end{array} \qquad \left(\begin{array}{rrrrcr} 1 & -2 & 1 & 1 & | & 0 \\ -1 & 0 & -3 & 1 & | & 0 \\ 1 & 3 & 6 & -4 & | & 0 \\ -1 & 0 & -3 & 1 & | & 0 \end{array}\right).$$

We can reduce this matrix to

$$\left(\begin{array}{rrrrcr} 1 & 0 & 3 & -1 & | & 0 \\ 0 & 1 & 1 & -1 & | & 0 \\ 0 & 0 & 0 & 0 & | & 0 \\ 0 & 0 & 0 & 0 & | & 0 \end{array}\right),$$

which is in reduced row–echelon form. We therefore get that the system in (5) is equivalent to the system

$$\left\{\begin{array}{rcl} c_1 + 3c_3 - c_4 & = & 0 \\ c_2 + c_3 - c_4 & = & 0. \end{array}\right. \tag{6}$$

Solving for the leading variables in (6) yields the solutions

$$\left\{\begin{array}{rcl} c_1 & = & 3t + s \\ c_2 & = & t + s \\ c_3 & = & -t \\ c_4 & = & s, \end{array}\right. \tag{7}$$

where t and s are arbitrary parameters. Taking $t = 1$ and $s = 0$ in (7) yields from (4) the linear relation

$$3v_1 + v_2 - v_3 = \mathbf{0},$$

which shows that $v_3 = -3v_1 - v_2$; that is, $v_3 \in \text{span}\{v_1, v_2\}$.

Similarly, taking $t = 0$ and $s = 1$ in (7) yields

$$v_1 + v_2 + v_4 = \mathbf{0},$$

which shows that $v_4 = -v_1 - v_2$; that is, $v_4 \in \text{span}\{v_1, v_2\}$.
We then have that both v_3 and v_4 are in the span of $\{v_1, v_2\}$. Consequently,

$$\{v_1, v_2, v_3, v_4\} \subseteq \text{span}\{v_1, v_2\},$$

from which we get that

$$\text{span}\{v_1, v_2, v_3, v_4\} \subseteq \text{span}\{v_1, v_2\},$$

since $\text{span}\{v_1, v_2, v_3, v_4\}$ is the smallest subspace of $\mathbb{R}^3$ which contains $\{v_1, v_2, v_3, v_4\}$. Combining this with

$$\text{span}\{v_1, v_2\} \subseteq \text{span}\{v_1, v_2, v_3, v_4\},$$

we conclude that

$$\text{span}\{v_1, v_2\} = \text{span}\{v_1, v_2, v_3, v_4\};$$

that is, $\{v_1, v_2\}$ spans $\text{span}\{v_1, v_2, v_3, v_4\}$.
To see that $\{v_1, v_2\}$ is linearly independent, observe that v_1 and v_2 are not multiples of each other. We therefore conclude that $\{v_1, v_2\}$ is a basis for $\text{span}\{v_1, v_2, v_3, v_4\}$. $\square$

3. Find a basis for the solution space of the system

$$\begin{cases} x_1 - x_2 + x_3 - x_4 = 0 \\ 2x_1 - x_2 - 2x_4 = 0 \\ -x_1 + x_3 + x_4 = 0, \end{cases} \tag{8}$$

and compute its dimension.

***Solution**:* We first find the solution space, W, of the system. In order to do this, we reduce the augmented matrix of this system,

$$\begin{array}{c} R_1 \\ R_2 \\ R_3 \end{array} \quad \left(\begin{array}{rrrr|r} 1 & -1 & 1 & -1 & 0 \\ 2 & -1 & 0 & -2 & 0 \\ -1 & 0 & 1 & 1 & 0 \end{array} \right),$$

to its reduced row–echelon form:

$$\left(\begin{array}{cccc|c} 1 & 0 & -1 & 0 & 0 \\ 0 & 1 & -2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right).$$

Consequently, the system in (8) is equivalent to the system

$$\left\{\begin{array}{rcrcl} x_1 & - & x_3 & = & 0 \\ & x_2 - & 2x_3 & = & 0 \\ & & x_4 & = & 0. \end{array}\right. \tag{9}$$

Solving for the leading variables in the system in (9) we obtain the solutions

$$\left\{\begin{array}{rcl} x_1 & = & t \\ x_2 & = & 2t \\ x_3 & = & t \\ x_4 & = & 0, \end{array}\right.$$

where t is an arbitrary parameter. I then follows that the solution space of system (9) is

$$W = \operatorname{span}\left\{\begin{pmatrix} 1 \\ 2 \\ 1 \\ 0 \end{pmatrix}\right\}.$$

Hence

$$\left\{\begin{pmatrix} 1 \\ 2 \\ 1 \\ 0 \end{pmatrix}\right\}$$

is a basis for W and therefore $\dim(W) = 1$. □

4. Prove that any set of four vectors in $\mathbb{R}^3$ must be linearly dependent.

Proof: Let v_1 , v_2, v_3 and v_4 denote four vectors in $\mathbb{R}^3$ and write

$$v_1 = \begin{pmatrix} a_{11} \\ a_{21} \\ a_{31} \end{pmatrix}, \quad v_2 = \begin{pmatrix} a_{12} \\ a_{22} \\ a_{32} \end{pmatrix}, \quad v_3 = \begin{pmatrix} a_{13} \\ a_{23} \\ a_{33} \end{pmatrix} \quad \text{and} \quad v_4 = \begin{pmatrix} a_{14} \\ a_{24} \\ a_{34} \end{pmatrix}.$$

Consider the vector equation

$$c_1v_1 + c_2v_2 + c_3v_3 + c_4v_4, = \mathbf{0}. \tag{10}$$

This equation translate into the homogeneous system

$$\left\{\begin{array}{rcl} a_{11}c_1 + a_{12}c_2 + a_{13}c_3 + a_{14}c_4 & = & 0 \\ a_{21}c_1 + a_{22}c_2 + a_{23}c_3 + a_{24}c_4 & = & 0 \\ a_{31}c_1 + a_{32}c_2 + a_{33}c_3 + a_{34}c_4 & = & 0, \end{array}\right. \tag{11}$$

of 3 linear equations in 4 unknowns. It then follows from the Fundamental Theorem for homogeneous linear systems that system (11) has infinitely many solutions. Consequently, the vector equation in (10) has a nontrivial solution, and therefore the set $\{v_1, v_2, v_3, v_4\}$ is linearly dependent. □

5. Show that if the set $\{v_1, v_2\}$ is a linearly independent subset of $\mathbb{R}^n$, then so is the set $\{v_1, cv_1 + v_2\}$, where c is a scalar, and, conversely, if $\{v_1, cv_1 + v_2\}$ is linearly independent, then so is $\{v_1, v_2\}$. Show also that $\text{span}\{v_1, v_2\} = \text{span}\{v_1, cv_1 + v_2\}$.

   (a) First we prove that $\{v_1, v_2\}$ is a linearly independent subset of $\mathbb{R}^n$, then so is the set $\{v_1, cv_1 + v_2\}$,

   Proof: Assume that $\{v_1, v_2\}$ is a linearly independent and consider the vector equation

   $$c_1v_1 + c_2(cv_1 + v_2) = \mathbf{0}. \tag{12}$$

   Applying the distributive and associative properties, the equation in (12) turns into

   $$(c_1 + cc_2)v_1 + c_2v_2 = \mathbf{0}. \tag{13}$$

   It follows from (13) and the linear independence of $\{v_1, v_2\}$ that

   $$\left\{\begin{array}{rcl} c_1 + cc_2 & = & 0 \\ c_2 & = & 0. \end{array}\right. \tag{14}$$

   The system in (14) has only the trivial solution: $c_2 = c_1 = 0$. Hence, the vector equation in (12) has only the trivial solution and therefore the set $\{v_1, cv_1 + v_2\}$ is linearly independent. □

   (b) Next, we prove the converse of the statement in (a): If $\{v_1, cv_1 + v_2\}$ is linearly independent, then $\{v_1, v_2\}$ is a linearly independent.

   Proof: Assume that $\{v_1, cv_1 + v_2\}$ is a linearly independent and consider the vector equation

   $$c_1v_1 + c_2v_2 = \mathbf{0}. \tag{15}$$

Adding $\mathbf{0} = cc_2v_1 - cc_2v_1$ to the left–hand side of the equation in (15) and applying the distributive and associative properties we get

$$(c_1 - cc_2)v_1 + c_2(cv_1 + v_2) = \mathbf{0}. \tag{16}$$

It follows from (16) and the linear independence of $\{v_1, cv_1 + v_2\}$ that

$$\left\{ \begin{array}{rcl} c_1 - cc_2 & = & 0 \\ c_2 & = & 0. \end{array} \right. \tag{17}$$

The system in (17) has only the trivial solution: $c_2 = c_1 = 0$. Hence, the vector equation in (15) has only the trivial solution and therefore the set $\{v_1, v_2\}$ is linearly independent. $\square$

(c) We prove that that $\text{span}\{v_1, v_2\} = \text{span}\{v_1, cv_1 + v_2\}$.

Proof: Let $W = \text{span}\{v_1, v_2\}$. Then, W is a subspace which contains v_1 and v_2 and all their linear combinations; in particular, $cv_1 + v_2 \in W$. We then have that

$$\{v_1, cv_1 + v_2\} \subseteq W.$$

It then follows that

$$\text{span}\{v_1, cv_1 + v_2\} \subseteq W, \tag{18}$$

since $\text{span}\{v_1, cv_1 + v_2\}$ is the smallest subspace of $\mathbb{R}^n$ which contains $\{v_1, cv_1 + v_2\}$. Ont the other hand, for any $u \in W$ there exist scalars c_1 and c_2 such that

$$u = c_1v_1 + c_2v_2.$$

Consequently,

$$\begin{array}{rcl} u & = & c_1v_1 + c_2v_2 + cc_2v_1 - cc_2v_1 \\ & = & (c_1 - cc_2)v_1 + c_2(cv_1 + v_2), \end{array}$$

which shows that $u \in \text{span}\{v_1, cv_1 + v_2\}$; thus,

$$u \in W \Rightarrow u \in \text{span}\{v_1, cv_1 + v_2\},$$

or

$$W \subseteq \text{span}\{v_1, cv_1 + v_2\}.$$

Combining this with (18) yields that

$$W = \text{span}\{v_1, cv_1 + v_2\}.$$

$\square$

6. Let J and H be planes in $\mathbb{R}^3$ given by

$$J = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \;\middle|\; 2x + 3y - 6z = 0 \right\} \quad \text{and} \quad H = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \;\middle|\; x - 2y + z = 0 \right\}.$$

(a) Give bases for J and H and compute their dimensions.

Solution: To find a basis for J, we solve the equation

$$2x + 3y + z = 0$$

to get the solution space $J = \operatorname{span}\left\{ \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix} \right\}$. Thus, the set

$$\left\{ \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 3 \\ 1 \end{pmatrix} \right\}$$

is a basis for J and so $\dim(J) = 2$.
Similarly, for H, we solve

$$x - 2y + z = 0$$

and obtain that

$$\left\{ \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \right\}$$

is a basis for H; thus, $\dim(H) = 2$. $\square$

(b) Give a basis for the subspace $J \cap H$ and compute $\dim(J \cap H)$.

Solution: Vectors $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ in the intersection of J and H if they are solutions to the system of equations

$$\left\{ \begin{array}{rcl} 2x + 3y - 6z & = & 0 \\ x - 2y + z & = & 0. \end{array} \right. \tag{19}$$

Thus, to find $J \cap H$, we may elementary row operations on the augmented matrix

$$\begin{array}{c} R_1 \\ R_2 \end{array} \quad \left(\begin{array}{rrr|r} 2 & 3 & -6 & 0 \\ 1 & -2 & 1 & 0 \end{array} \right)$$

to obtain the reduced matrix

$$\left(\begin{array}{ccc|c} 1 & 0 & -9/7 & 0 \\ 0 & 1 & -8/7 & 0 \end{array}\right).$$

Thus, the system in (19) is equivalent to

$$\left\{\begin{array}{rcl} x \;-\; \frac{9}{7}z & = & 0 \\ y - \frac{8}{7}z & = & 0, \end{array}\right. \tag{20}$$

Solving for the leading variables in system (20) and setting $z = 7t$, where t is an arbitrary parameter, wee obtain that

$$J \cap H = \operatorname{span}\left\{\begin{pmatrix} 9 \\ 8 \\ 7 \end{pmatrix}\right\}.$$

Thus, the set

$$\left\{\begin{pmatrix} 9 \\ 8 \\ 7 \end{pmatrix}\right\}$$

is a basis for $J \cap H$ and, therefore, $\dim(J \cap H) = 1$. □

7. Let W be a subspace of $\mathbb{R}^n$.

   (a) Prove that if $v \in W$ and $v \neq \mathbf{0}$, then $rv = sv$ implies that $r = s$, where r and s are scalars.

   Proof: Suppose that $v \in W$, where W is a subspace of $\mathbb{R}^n$, and that $v \neq \mathbf{0}$. Suppose also that

$$rv = sv \tag{21}$$

   for some scalars r and s. Add $-sv$ on both sides of the vector equation in (21) and apply the distributive property to obtain

$$(r - s)v = \mathbf{0}. \tag{22}$$

   Taking the Euclidean inner product with v of both sides of (22) yields

$$(r - s)\langle v, v\rangle = 0, \tag{23}$$

   where we have used the bi–linearity of the inner product. It then follows form (23), the positive definiteness of the inner product, and the assumption that $v \neq \mathbf{0}$, that

$$r - s = 0$$

   and therefore $r = s$, which was to be shown. □

(b) Prove that if W has more than one element, then W has infinitely many elements.

Proof: Since W has at least two elements, there has to be a vector, v, in W such that $v \neq \mathbf{0}$. Now, for any $t \in \mathbb{R}$, $tv \in W$ because W is closed under scalar multiplication. By part (a), $t_1 v \neq t_2 v$ for any $t_1 \neq t_2$. Consequently, W contains infinitely many vectors. $\square$

8. Let W be a subspace of $\mathbb{R}^n$ and S_1 and S_2 be subsets of W.

(a) Show that $\operatorname{span}(S_1 \cap S_2) \subseteq \operatorname{span}(S_1) \cap \operatorname{span}(S_2)$.

Proof: First observe that $S_1 \cap S_2 \subseteq S_1$ and $S_1 \cap S_2 \subseteq S_2$. Consequently,

$$\operatorname{span}(S_1 \cap S_2) \subseteq \operatorname{span}(S_1) \quad \text{and} \quad \operatorname{span}(S_1 \cap S_2) \subseteq \operatorname{span}(S_2).$$

It then follows that

$$\operatorname{span}(S_1 \cap S_2) \subseteq \operatorname{span}(S_1) \cap \operatorname{span}(S_2),$$

which was to be shown. $\square$

(b) Give an example in which $\operatorname{span}(S_1 \cap S_2) \neq \operatorname{span}(S_1) \cap \operatorname{span}(S_2)$.

***Solution**:* Let $S_1 = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\}$ and $S_2 = \left\{ \begin{pmatrix} -1 \\ 0 \end{pmatrix} \right\}$. Then, $S_1 \cap S_2 = \emptyset$ so that $\operatorname{span}(S_1 \cap S_2) = \{\mathbf{0}\}$, where $\mathbf{0}$ denotes the zero vector in $\mathbb{R}^2$.
On the other hand,

$$\operatorname{span}(S_1) = \left\{ \begin{pmatrix} x \\ 0 \end{pmatrix} \in \mathbb{R}^2 \;\middle|\; x \in \mathbb{R} \right\}$$

and

$$\operatorname{span}(S_2) = \left\{ \begin{pmatrix} x \\ 0 \end{pmatrix} \in \mathbb{R}^2 \;\middle|\; x \in \mathbb{R} \right\}.$$

Hence,

$$\operatorname{span}(S_1) \cap \operatorname{span}(S_2) = \left\{ \begin{pmatrix} x \\ 0 \end{pmatrix} \in \mathbb{R}^2 \;\middle|\; x \in \mathbb{R} \right\} \neq \{\mathbf{0}\}.$$

$\square$

(c) Show that if $S_1 \subseteq S_2$ and S_2 is linearly independent, then S_1 is also linearly independent.

Proof: Suppose that $S_1 \subseteq S_2$ and S_2 is linearly independent, and that $c_1, c_2, \ldots, c_n$ solve the vector equation

$$c_1 v_1 + c_2 v_2 + \cdots + c_k v_k = \mathbf{0}, \tag{24}$$

where $v_1, v_2, \ldots, v_k$ are vectors in S_1. Since we are assuming that $S_1 \subseteq S_2$, the vectors $v_1, v_2, \ldots, v_k$ are also in S_2, which is assumed to be linearly independent; consequently,

$$c_1 = c_2 = \cdots = c_k = 0.$$

Thus, we have shown that for any finite set of vectors, $v_1, v_2, \ldots, v_k$, in S_1, the vector equation in (24) has only the trivial solution. Hence, S_1 is linearly independent. $\square$

(d) Show that if $S_1 \subseteq S_2$ and S_1 is linearly dependent, then S_2 is also linearly dependent.

Proof: Suppose that $S_1 \subseteq S_2$ and S_1 is linearly dependent. Then, there exist vectors $v_1, v_2, \ldots, v_k$ are vectors in S_1 such that the equation

$$c_1 v_1 + c_2 v_2 + \cdots + c_k v_k = \mathbf{0}, \tag{25}$$

has a nontrivial solution. Since we are assuming that $S_1 \subseteq S_2$, the vectors $v_1, v_2, \ldots, v_k$ are also in S_2. This proves that S_2 is linearly dependent. $\square$

9. Let W_1 and W_2 be two subspaces of $\mathbb{R}^n$. We write $W_1 \oplus W_2$ for the subspace $W_1 + W_2$ for the special case in which $V = W_1 \cap W_2 = \{\mathbf{0}\}$. Show that every vector $v \in W_1 \oplus W_2$ can be written in the form $v = v_1 + v_2$, where $v_1 \in W_1$ and $v_2 \in W_2$, in one and only one way; that is, if $v = u_1 + u_2$, where $u_1 \in W_1$ and $u_2 \in W_2$, then $u_1 = v_1$ and $u_2 = v_2$.

Proof: Suppose that W_1 and W_2 are two subspaces of $\mathbb{R}^n$ which have only the zero vector in common; that is, $W_1 \cap W_2 = \{\mathbf{0}\}$. Let v be any in $\in W_1 + W_2$. Then, $v = v_1 + v_2$, where $v_1 \in W_1$ and $v_2 \in W_2$. Suppose that v can also be written as $v = u_1 + u_2$, where $u_1 \in W_1$ and $u_2 \in W_2$. Then,

$$v_1 + v_2 = u_1 + u_2,$$

from which we get that

$$v_1 - u_1 = v_2 - u_2, \tag{26}$$

where $v_1 - u_1 \in W_1$ and $v_2 - u_2 \in W_2$ since W_1 and W_2 are subspaces of $\mathbb{R}^n$. It also follows from (26) that $v_1 - u_1 \in W_2$. Thus, $v_1 - u_1 \in W_1 \cap W_2 = \{\mathbf{0}\}$, which implies that

$$v_1 - u_1 = \mathbf{0},$$

or

$$v_1 = u_1.$$

Similarly, we get that $v_2 = u_2$. $\square$

10. Let $v \in \mathbb{R}^n$ and define $W = \{w \in \mathbb{R}^n \mid \langle w, v\rangle = 0\}$.

   (a) Prove that W is a subspace of $\mathbb{R}^n$.

   Proof: First, observe that $W \neq \emptyset$ because $\langle \mathbf{0}, v\rangle = 0$ and therefore $\mathbf{0} \in W$ and so W is nonempty.
   Next, we show that W is closed under addition and scalar multiplication.
   To see that W is closed under scalar multiplication, observe that, by the bi–linearity property of the inner product, if $w \in W$, then

$$\langle\langle v, tw\rangle = t\langle v, w\rangle = t \cdot 0 = 0$$

   for all $t \in \mathbb{R}$.
   To show that W is closed under vector addition, let w_1 and w_2 be two vectors in W. Then, applying the bi–linearity property of the inner product again,

$$\langle w_1 + w_2, v\rangle = \langle w_1, v\rangle + \langle w_2, v\rangle = 0 + 0 = 0;$$

   hence, $w_1 + w_2 \in W$. $\square$

   (b) Suppose that $v \neq \mathbf{0}$ and compute $\dim(W)$.

   Solution: Let $B = \{w_1, w_2, \ldots w_k\}$ be a basis for W. Then, $\dim(W) = k$ and we would like to determine what k is.
   First note that $v \notin \operatorname{span}(B)$. For, suppose that $v \in \operatorname{span}(B) = W$, then

$$\langle v, v\rangle = 0.$$

   Thus, by the positive definiteness of the Euclidean inner product, it follows that $v = \mathbf{0}$, but we are assuming that $v \neq \mathbf{0}$. Consequently, the set

$$B \cup \{v\} = \{w_1, w_2, \ldots w_k, v\}$$

is linearly independent. We claim that $B \cup \{v\}$ also spans $\mathbb{R}^n$. To see why this is so, let $u \in \mathbb{R}^n$ be any vector in $\mathbb{R}^n$, and let

$$t = \frac{\langle u, v\rangle}{\|v\|^2}.$$

Write

$$u = tv + (u - tv),$$

and observe that $u - tv \in W$. To see why this is so, compute

$$\begin{aligned}
\langle u - tv, v\rangle &= \langle u, v\rangle - t\langle v, v\rangle \\
&= \langle u, v\rangle - t\|v\|^2 \\
&= \langle u, v\rangle - \frac{\langle u, v\rangle}{\|v\|^2}\|v\|^2 \\
&= \langle u, v\rangle - \langle u, v\rangle \\
&= 0.
\end{aligned}$$

Thus, $u - tv \in W$. It then follows that there exist scalars $c_1, c_2, \ldots, c_k$ such that

$$u - tv = c_1 w_1 + c_2 w_2 + \cdots + c_k w_k.$$

Thus,

$$u = c_1 w_1 + c_2 w_2 + \cdots + c_k w_k + tv,$$

which shows that $u \in \operatorname{span}(B \cup \{v\})$. Consequently, $B \cup \{v\}$ spans $\mathbb{R}^n$. Therefore, since $B \cup \{v\}$ is also linearly independent, it forms a basis for $\mathbb{R}^n$. We then have that $B \cup \{v\}$ must have n vectors in it, since $\dim(\mathbb{R}^n) = n$; that is,

$$k + 1 = n,$$

from which we get that

$$\dim(W) = n - 1.$$

$\square$