Solutions to Exam 1

1. Answer the following questions as thoroughly as possible.

   (a) State precisely what it means for the set of vectors \( \{v_1, v_2, \ldots, v_k\} \) in \( \mathbb{R}^n \) to be linearly independent.

   \textbf{Answer:} The set \( \{v_1, v_2, \ldots, v_k\} \) is linearly independent in \( \mathbb{R}^n \) if and only if the vector equation
   \[ c_1v_1 + c_2v_2 + \cdots + c_kv_k = 0 \]
   has only the trivial solution
   \[ c_1 = c_2 = \cdots = c_k = 0. \]

   \[ \square \]

   \textbf{Alternate Answer:} The set \( \{v_1, v_2, \ldots, v_k\} \) is linearly independent in \( \mathbb{R}^n \) if and only if no vector in the set is in the span of the other vectors.

   \[ \square \]

   (b) Define the span of the set of vectors, \( S \), in \( \mathbb{R}^n \).

   \textbf{Answer:} The span of \( S \) is the set of all finite linear combinations of vectors in \( S \).

   \[ \square \]

   \textbf{Alternate Answer:} The span of \( S \) is the smallest subspace of \( \mathbb{R}^n \) which contains \( S \).

   \[ \square \]

   (c) Let \( W \) denote a subspace of \( \mathbb{R}^n \). Define the coordinates of a vector \( v \in W \) relative to a basis \( B \) for \( W \).

   \textbf{Answer:} Let \( B = \{w_1 + w_2 + \cdots + w_k\} \) be an ordered basis for \( W \). Given \( v \in W \), the coordinates of \( v \) relative to \( B \) are the unique set of scalars \( c_1, c_2, \ldots, c_k \) such that
   \[ v = c_1w_1 + c_2w_2 + \cdots + c_kw_k. \]

   \[ \square \]

2. Determine whether the following statements are true or false. If false, give examples to justify your conclusion. If true, provide an argument to justify your answer.
(a) The set, \( \{v_1, v_2, v_3\} \), of vectors in \( \mathbb{R}^2 \) is linearly dependent.

**Answer:** True.

**Proof:** Write
\[
v_1 = \begin{pmatrix} a_1 \\ b_1 \end{pmatrix}, \quad v_2 = \begin{pmatrix} a_2 \\ b_2 \end{pmatrix}, \quad \text{and} \quad v_3 = \begin{pmatrix} a_3 \\ b_3 \end{pmatrix},
\]
and consider the vector equation
\[
c_1v_1 + c_2v_2 + c_3v_3 = 0,
\]
where \( 0 \) denotes the zero–vector in \( \mathbb{R}^2 \). Equation (1) is equivalent to the system
\[
\begin{align*}
a_1c_1 + a_2c_2 + a_3c_3 &= 0 \\
b_1c_1 + b_2c_2 + b_3c_3 &= 0,
\end{align*}
\]
which is a homogeneous system of two linear equations in three unknowns. It follows by the Fundamental Theorem of Homogeneous Linear Systems that (2) has a nontrivial solution. Consequently, the vector equation in (1) has a nontrivial solution and therefore \( \{v_1, v_2, v_3\} \) is a linearly dependent subset of \( \mathbb{R}^2 \).

(b) The set of vectors in \( \mathbb{R}^3 \), \( \{0, v_1, v_2\} \) is linearly independent.

**Answer:** False.

Note that \( 0 = 0 \cdot v_1 + 0 \cdot v_2 \), and so \( 0 \) is in the span of \( v_1 \) and \( v_2 \).

(c) If \( S_1 \) and \( S_2 \) are linearly independent, then \( S_1 \cup S_2 \) is also linearly independent.

**Answer:** False.

Let \( S_1 = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\} \) and \( S_2 = \left\{ \begin{pmatrix} 2 \\ 0 \end{pmatrix} \right\} \). Then, \( S_1 \) and \( S_2 \) are linearly independent, but
\[
S_1 \cup S_2 = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \end{pmatrix} \right\}
\]
is linearly dependent since \( \begin{pmatrix} 2 \\ 0 \end{pmatrix} \) is a scalar multiple of \( \begin{pmatrix} 1 \\ 0 \end{pmatrix} \).
3. Let \( \langle v, w \rangle \) denote the Euclidean inner product in \( \mathbb{R}^n \). For a fixed vector \( u \) in \( \mathbb{R}^n \), define the set
\[
W = \{ w \in \mathbb{R}^n \mid \langle u, w \rangle = 0 \}.
\]
Prove that \( W \) is a subspace of \( \mathbb{R}^n \).

**Proof:** First, observe that \( W \neq \emptyset \) because \( \langle u, 0 \rangle = 0 \) and therefore \( 0 \in W \) and so \( W \) is nonempty.

Next, we show that \( W \) is closed under addition and scalar multiplication.

To see that \( W \) is closed under scalar multiplication, observe that, by the bi–linearity property of the inner product, if \( w \in W \), then
\[
\langle u, tw \rangle = t \langle u, w \rangle = t \cdot 0 = 0
\]
for all \( t \in \mathbb{R} \).

To show that \( W \) is closed under vector addition, let \( w_1 \) and \( w_2 \) be two vectors in \( W \). Then, applying the bi–linearity property of the inner product again,
\[
\langle v, w_1 + w_2 \rangle = \langle u, w_1 \rangle + \langle u, w_2 \rangle = 0 + 0 = 0;
\]
hence, \( w_1 + w_2 \in W \). \( \square \)

4. Find a basis for the solution space, \( W \), of the homogenous system
\[
\begin{align*}
3x_1 - x_2 + 2x_3 + x_4 &= 0 \\
2x_1 - x_2 + x_3 &= 0 \\
x_1 + x_3 + x_4 &= 0,
\end{align*}
\]
and compute \( \dim(W) \).

**Solution:** We first find the solution space, \( W \), of the system. In order to do this, we reduce the augmented matrix of this system,
\[
\begin{pmatrix}
R_1 & 3 & -1 & 2 & 1 & 0 \\
R_2 & 2 & -1 & 1 & 0 & 0 \\
R_3 & 1 & 0 & 1 & 1 & 1
\end{pmatrix},
\]
to its reduced row–echelon form:
\[
\begin{pmatrix}
1 & 0 & 1 & 1 & 0 \\
0 & 1 & 1 & 2 & 0 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix}.
\]
Consequently, the system in (3) is equivalent to the system

\[
\begin{align*}
    x_1 + x_3 + x_4 &= 0 \\
    x_2 + x_3 + 2x_4 &= 0.
\end{align*}
\]  

(4)

Solving for the leading variables in the system in (4) and setting \(x_3 = -t\) and \(x_4 = -s\), where \(t\) and \(s\) are arbitrary parameters, we obtain the solutions

\[
\begin{align*}
    x_1 &= t + s \\
    x_2 &= t + 2s \\
    x_3 &= -t \\
    x_4 &= -s.
\end{align*}
\]

It then follows that the solution space of system (4) is

\[
W = \text{span}\left\{ \begin{pmatrix} 1 \\ 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 0 \\ -1 \end{pmatrix} \right\}.
\]

Hence, the set

\[
B = \left\{ \begin{pmatrix} 1 \\ 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 0 \\ -1 \end{pmatrix} \right\}
\]

spans \(W\). It is also linearly independent since the vectors in \(B\) are not multiples of each other. Consequently, \(B\) is a basis for \(W\) and therefore \(\dim(W) = 2\). \qed