Review Problems for Exam 2

1. Let u denote a unit vector in $\mathbb{R}^n$ and define $f\colon \mathbb{R}^n \to \mathbb{R}^n$ by

$$f(v) = \langle u, v\rangle u \quad \text{for all } v \in \mathbb{R}^n,$$

where $\langle\cdot,\cdot\rangle$ denotes the Euclidean inner product in $\mathbb{R}^n$.

   (a) Verify that f is linear.

   (b) Give the image, $\mathcal{I}_f$, and null space, $\mathcal{N}_f$, of f, and compute $\dim(\mathcal{I}_f)$.

   (c) Use the Dimension Theorem for linear transformations $T\colon \mathbb{R}^n \to \mathbb{R}^m$,

$$\dim(\mathcal{N}_T) + \dim(\mathcal{I}_T) = n,$$

   to compute $\dim(\mathcal{N}_f)$.

2. Let $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ denote the linear transformation which maps the parallelogram spanned by

$$v_1 = \begin{pmatrix} 2 \\ -1 \end{pmatrix} \quad \text{and} \quad v_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

to the parallelogram spanned by

$$w_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix} \quad \text{and} \quad w_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

   (a) Give the matrix representation, M_T, relative to the standard basis in $\mathbb{R}^2$.

   (b) Compute $\det(T)$. Does T preserve orientation?

   (c) Show that T is invertible and compute the inverse of T.

   (d) Does T have real eigenvalues? If so, compute them and their corresponding eigenspaces.

3. Define $T\colon \mathbb{R}^3 \to \mathbb{R}^3$ by

$$T(v) = Av \quad \text{for all } v \in \mathbb{R}^3,$$

where A is the 3×3 matrix given by

$$A = \begin{pmatrix} 1 & 2 & 1 \\ 6 & -1 & 0 \\ -1 & -2 & -1 \end{pmatrix}.$$

Find all eigenvalues and corresponding eigenspaces for the transformation T.

4. Find a value of d for which the matrix

$$A = \begin{pmatrix} 1 & -2 \\ 3 & d \end{pmatrix}$$

   is not invertible.

   Show that, for that value of d, $\lambda = 0$ is an eigenvalue of A. Give the eigenspace corresponding to 0. What is the dimension of $E_A(0)$?

5. Use the fact that $\det(AB) = \det(A)\det(B)$ for all $A, B \in \mathbb{M}(n,n)$ to compute $\det(A^{-1}$, provided that A is invertible.

6. Let A and B be $n \times n$ matrices. Show that if AB is invertible, then so is A.

7. Let A be a 3×3 matrix satisfying $A^3 - 6A^2 - 2A + 12I = O$, where I is the 3×3 identity matrix and O is the 3×3 zero matrix.

   (a) Prove that A is invertible and given a formula for computing its inverse in terms of I, A and A^2.

   (b) Prove that if λ is an eigenvalue of A, then $\lambda^3 - 6\lambda^2 - 2\lambda + 12 = 0$. Deduce therefore that λ is one of 6, $\sqrt{2}$ or $-\sqrt{2}$.

8. Let $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ be given by $T(v) = Av$ for all $v \in \mathbb{R}^2$, where A is a 2×2 matrix. Let $\text{area}(P(v_1, v_2))$ denote the are of the parallelogram determined by the vectors v_1 and v_2. Prove that

$$\text{area}P((T(v_1), T(v_2))) = |\det(A)| \cdot \text{area}(P(v_1, v_2)).$$