Topics for Final Exam

1. **Vector Space Structure in Euclidean Space**

   1.1 Definition of n–Dimensional Euclidean Space

   1.2 Vector addition and scalar multiplication

   1.3 Spans

   1.4 Linear independence

2. **Subspaces of Euclidean Space**

   2.1 Bases

   2.2 Dimension

   2.3 Coordinates

3. **Connections with the Theory of Linear Equations**

   3.1 Homogeneous systems

   3.2 Fundamental Theorem for homogenous systems of linear equations

   3.3 Nonhomogeneous systems

4. **Euclidean Inner Product and Norm**

   4.1 Row–column product

   4.2 Euclidean inner product

   4.3 Euclidean norm

   4.4 Orthogonality

   4.5 Orthonormal bases

5. **Spaces of Matrices**

   5.1 Matrix Algebra

   5.2 Null space and nullity

   5.3 Column and row spaces

   5.4 Rank

   5.5 Invertibility

6. **Linear Transformations**

    6.1 Definition of linearity

    6.2 Matrix representation

    6.3 Null space and image

    6.4 Compositions

    6.5 Invertible linear transformation

    6.6 Orthogonal transformations

        6.6.1 Orthogonal matrices
        6.6.2 Determinant, cross–product and triple–scalar product
        6.6.3 Areas and volumes
        6.6.4 Area and volume preserving transformations
        6.6.5 Orientation
        6.6.6 Orientation preserving transformations

7. **The Eigenvalue Problem**

    7.1 Eigenvalues, eigenvectors and eigenspaces

    7.2 The eigenvalue problem

    7.3 Invariant subspaces

    7.4 Orthogonal, orientation reversing transformations in $\mathbf{R}^2$

    7.5 Orthogonal, orientation preserving transformations in $\mathbf{R}^3$

Relevant sections in text: 1.5, 1.6, 1.8, 2.2, 2.3, 3.2, 3.3, 3.4, 3.5, 3.6, 4.1, 4.3, 4.4, 5.1, 5.2, 6.1, 6.2, 6.3, 7.2 and 8.1.

Relevant chapters in the online class notes: Chapters 2, 3, 4, and 5.

Important Concepts: Euclidean space, linear independence, span, subspaces, bases, dimension, coordinates, inner product, norm, orthogonality, linear transformation, null space, image, nullity, rank, elementary matrices, invertibility, eigenvalue, eigenvector and eigenspace.

Important Results

Fundamental Theorem of Homogeneous Linear Systems. A homogeneous system of m linear equations in n unknowns,

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n & = & 0 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n & = & 0 \\ \vdots & \vdots & \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n & = & 0, \end{cases}$$

with $n > m$, has infinitely many solutions.

Equality of row rank and column rank. Let $A \in \mathbf{M}(m, n)$ and denote by $\mathcal{R}_A$ and $\mathcal{C}_A$ the row–space and column–space of A, respectively. Then,

$$\dim(\mathcal{R}_A) = \dim(\mathcal{C}_A).$$

Dimension Theorem for Linear Transformations. Let $T\colon \mathbf{R}^n \to \mathbf{R}^m$ be a linear transformation and denote by $\mathcal{N}_T$ and $\mathcal{I}_T$ the null–space and image of T, respectively. Then,

$$\dim(\mathcal{N}_T) + \dim(\mathcal{I}_T) = n.$$

Important Skills

Know how to determine whether subsets of $\mathbf{R}^n$ are linearly independent; know how to tell whether a given subset of $\mathbf{R}^n$ is a subspace; know how to tell whether a set of vectors in $\mathbf{R}^n$ spans a subspace; know how to compute the span of a set of vectors; know how to solve systems of linear equations; know how to determine bases for subspaces of Euclidean space; know how to compute dimensions of subspaces; know how to find coordinates of vectors relative to ordered bases; know how to tell whether vectors are orthogonal; know how to tell whether a given matrix is invertible or not; know how to compute inverses of invertible matrices; know how to determine whether a given function is linear or not; know how to obtain matrix representations of linear transformations; know how to compute nullity and rank of matrices, know how to compute determinants of 2×2 and 3×3 matrices; know how to find eigenvalues, eigenvectors and eigenspaces of linear transformations.