Solutions to Assignment #11

1. Use the fact that $\sqrt{2} = \sup\{q \in \mathbb{Q} \mid q > 0 \text{ and } q^2 < 2\}$ to prove that there exists a sequence of rational numbers, $(q_n)$, such that

$$\lim_{n \to \infty} q_n = \sqrt{2}.$$ 

*Proof:* Write $A = \{q \in \mathbb{Q} \mid q > 0 \text{ and } q^2 < 2\}$. Then, $\sqrt{2} = \sup(A)$. Thus, for each $n \in \mathbb{N}$ there exists $q_n \in A$ such that

$$\sqrt{2} - \frac{1}{n} < q_n < \sqrt{2}.$$

Since $\lim_{n \to \infty} \frac{1}{n} = 0$, it follows by the Squeeze Theorem for sequences that the limit of $(q_n)$ exists and

$$\lim_{n \to \infty} q_n = \sqrt{2}. \quad \square$$

2. Let $(\varepsilon_n)$ denote a sequence of positive numbers which converges to 0. Let $(x_n)$ be a sequence of real numbers and $x \in \mathbb{R}$. Assume there exists $N_1 \in \mathbb{N}$ such that

$$|x_n - x| \leq \varepsilon_n \quad \text{for all } n \geq N_1.$$ 

Prove that $(x_n)$ converges to $x$.

*Proof:* Assume there exists $N_1 \in \mathbb{N}$ such that

$$|x_n - x| \leq \varepsilon_n \quad \text{for all } n \geq N_1, \quad (1)$$

where $\varepsilon_n > 0$ for all $n$ and $\lim_{n \to \infty} \varepsilon_n = 0$.

Let $\varepsilon > 0$ be given. Then, there exists $N_2 \in \mathbb{N}$ such that

$$n \geq N_2 \Rightarrow \varepsilon_n < \varepsilon.$$ 

It then follows form (1) that, for $N = \max\{N_1, N_2\}$,

$$n \geq N \Rightarrow |x_n - x| < \varepsilon,$$

which shows that $(x_n)$ converges to $x$. \square
3. Let \( x_n = \frac{1}{n!} \) for \( n \in \mathbb{N} \). Prove that the sequence \((x_n)\) converges to 0.

**Proof:** First we prove by induction on \( n \) that \( n! \geq n \) for all \( n \in \mathbb{N} \). Note that \( 1! = 1 \) and so the result is true for the base step in the induction. Next, assume that \( n! \geq n \) and consider

\[
(n + 1)! = (n + 1)n! = n \cdot n! + n!.
\]

By the inductive hypothesis,

\[
(n + 1)! \geq n \cdot n + n \geq n + 1.
\]

We then have that

\[
0 < \frac{1}{n!} \leq \frac{1}{n}
\]

for all \( n \in \mathbb{N} \).

It then follows by the Squeeze Theorem, or by the result in the previous problem that

\[
\lim_{n \to \infty} \frac{1}{n!} = 0.
\]

4. Let \((x_n)\) be a sequence of real numbers converging to \( a \neq 0 \). Prove that there exists \( N \in \mathbb{N} \) such that

\[
n \geq N \Rightarrow |x_n| > \frac{|a|}{2}.
\]

**Proof.** Assume that \( \lim_{n \to \infty} x_n = a \), where \( a \neq 0 \). Put \( \varepsilon = \frac{|a|}{2} \). Then, \( \varepsilon > 0 \) and so, by the definition of convergence, there exists \( N \in \mathbb{N} \) such that

\[
n \geq N \Rightarrow |x_n - a| < \varepsilon = \frac{|a|}{2}.
\]

Now, by the triangle inequality

\[
|a| = |a - x_n + x_n| \leq |x_n - a| + |x_n|,
\]

for \( n \geq N \),

from which we get that

\[
|a| < \frac{|a|}{2} + |x_n|,
\]

for \( n \geq N \).

The result then follows by adding \( -\frac{|a|}{2} \) on both sides of the previous inequality. \( \square \)
5. Let \((x_n)\) be a sequence of non-zero, real numbers converging to \(a \neq 0\). Prove that the set \(A = \left\{ \frac{1}{x_n} \mid n \in \mathbb{N} \right\}\) is bounded.

Proof: Let \(x_n \neq 0\) for all \(n \in \mathbb{N}\) and assume that

\[
\lim_{n \to \infty} x_n = a, \quad \text{where} \quad a \neq 0.
\]

Then, by the result in the previous problem, there exists \(N \in \mathbb{N}\) such that

\[
\left| x_n \right| > \frac{|a|}{2} \quad \text{for all} \quad n \geq N.
\]

We then have that

\[
\frac{1}{\left| x_n \right|} < \frac{2}{|a|} \quad \text{for all} \quad n \geq N.
\]

Setting \(M = \max \left\{ \frac{1}{|x_1|}, \frac{1}{|x_2|}, \ldots, \frac{1}{|x_{N-1}|}, \frac{2}{|a|} \right\}\), we see that

\[
\frac{1}{\left| x_n \right|} \leq M \quad \text{for all} \quad n \in \mathbb{N};
\]

that is, the set \(A = \left\{ \frac{1}{x_n} \mid n \in \mathbb{N} \right\}\) is bounded. \(\square\)