Solutions to Assignment #3

1. Let \( x \) denote a real number satisfying \( x^2 = x \). Prove that either \( x = 0 \) or \( x = 1 \).
   (Note that \( x^2 = xx \).)

   \textit{Proof:} Let \( x \in \mathbb{R} \) and assume that
   \[ x^2 = x. \]
   Subtracting the additive inverse of \( x \), namely \(-x\), on both sides we obtain that
   \[ x^2 - x = 0, \]
   or
   \[ x(x - 1) = 0, \]
   (1)
   where we have used the distributive property (Axiom \((F_{10})\) in Handout #2). If \( x \neq 0 \), it follows from Axiom \((F_9)\) that there exists \( x^{-1} \in \mathbb{R} \) such that
   \[ x^{-1}x = 1. \]
   Multiplying on the left by \( x^{-1} \) on both sides of equation (1) we obtain
   \[ x^{-1}[x(x - 1)] = x^{-1}0, \]
   or
   \[ x - 1 = 0, \]
   (2)
   were we have used Axioms \((F_7), (F_9), (F_8)\) and the fact that \( a0 = 0 \) for all \( a \in \mathbb{R} \). Adding 1 on both sides of (2) yields
   \[ x = 1. \]
   Thus, we have shown that \( x^2 = x \) and \( x \neq 0 \) implies that \( x = 1 \), which is equivalent to \( x^2 = x \) implies \( x = 0 \) or \( x = 1 \). \( \square \)

2. Let \( a \in \mathbb{R} \). Prove that if \( a \neq 0 \), then the equation
   \[ ax = b \]
   has a unique solution for every \( b \in \mathbb{R} \).
Proof: Let $a \in \mathbb{R}$ and assume that $a \neq 0$. Then, by Axiom (F_9), there exists $a^{-1} \in \mathbb{R}$ such that $a^{-1}a = 1$. Let $x = a^{-1}b$. Then, by Axioms (F_7), (F_9), (F_8) and (F_6),

$$ax = a(a^{-1}b) = b,$$

which shows that $x = a^{-1}b$ is a solution of the equation

$$ax = b.$$  

To show that $ax = b$ has a unique solution, assume that $x_1$ and $x_2$ are two solutions of $ax = b$. Then,

$$ax_1 = b$$

and

$$ax_2 = b.$$

Consequently,

$$ax_1 = ax_2 \quad (3)$$

Multiplying both sides of equation (3) by $a^{-1}$ yields, by Axioms (F_7), (F_9), (F_8) and (F_6),

$$x_1 = x_2,$$

which shows that $ax = b$ has at most one solution.

3. Let $x \in \mathbb{R}$. Prove that $(-1)x$ is the additive inverse of $x$; that is $x + (-1)x = 0$.

Proof. Let $x \in \mathbb{R}$. Use Axioms to compute

$$x + (-1)x = 1x + (-1)x$$

$$= (1 + (-1))x$$

$$= 0x$$

$$= 0,$$

where we have used the fact that $0x = 0$ for all real numbers $x$.  

4. Prove that, for any real number, $x$,

$$(-x)^2 = x^2.$$
Proof: Let $x \in \mathbb{R}$. Using the fact that $(-1)(-x) = x$, where $-x$ is the additive inverse of $x$, and the associative property of multiplication we find that

$$x^2 = xx$$

$$= [(-1)(-x)][(-1)(-x)]$$

$$= (-1)(-1)(-x)(-x)$$

$$= 1(-x)^2$$

$$= (-x)^2,$$

which was to be shown.

5. Let $a, b \in \mathbb{Q}$, where $a^2 + b^2 \neq 0$.

(a) Explain by $a^2 - 2b^2 \neq 0$.

Solution: Since $a^2 + b^2 \neq 0$, if $b = 0$, then $a \neq 0$ and so $a^2 - 2b^2 = a^2 \neq 0$ in this case. Thus, we may assume that $b \neq 0$. Then, if $a^2 - 2b^2 = 0$, we have that

$$\frac{a^2}{b^2} = 2,$$

or

$$\left(\frac{a}{b}\right)^2 = 2,$$

which shows that there is $q \in \mathbb{Q}$ such that $q^2 = 2$; namely, $q = \frac{a}{b}$, since $a, b \in \mathbb{Q}$. This is impossible. Hence, $a^2 - 2b^2 \neq 0$, if $a^2 + b^2 \neq 0$. \qed

(b) Show that the multiplicative inverse of $a + b\sqrt{2}$, namely $(a + b\sqrt{2})^{-1}$, is of the form $c + d\sqrt{2}$, where $c, d \in \mathbb{Q}$.

Solution: Since $a^2 - 2b^2 \neq 0$, by part (a), we may define rational numbers

$$c = \frac{a}{a^2 - 2b^2} \quad \text{and} \quad d = \frac{-b}{a^2 - 2b^2},$$

since $a, b \in \mathbb{Q}$. \qed
Using the distributive property we may compute

\[(a + b\sqrt{2})(c + d\sqrt{2}) = \frac{1}{a^2 - 2b^2}(a + b\sqrt{2})(a - b\sqrt{2})\]

\[= \frac{1}{a^2 - 2b^2}(a^2 - (b\sqrt{2})^2)\]

\[= \frac{1}{a^2 - 2b^2}(a^2 - 2b^2)\]

\[= 1,\]

which shows that \(c + d\sqrt{2}\) is the multiplicative inverse of \(a + b\sqrt{2}\). □