Solutions to Assignment #5

1. Let $a$, $b$, $c$ and $d$ denote real numbers.  
Prove that if $a < b$ and $c < d$, then $a + c < b + d$.

   \textit{Proof:} Assume that $a < b$ and $c < d$. Then, by the definition of order in $\mathbb{R}$,  
   \[ b - a > 0 \quad \text{and} \quad d - c > 0. \]

   It then follows from Axiom $O_2$ that  
   \[ b - a + d - c > 0, \]

   where we have used the associative property of addition. Thus, using associativity of addition again, commutativity of addition and the distributive property,  
   we get that  
   \[ b + d - (a + c) > 0, \]

   which shows that  
   \[ a + c < b + d. \]

\[\square\]

2. For any real number $a$, show that $| - a | = |a|$.

   \textit{Proof:} Suppose first that $a > 0$. Then, $-a < 0$, so that  
   \[ |-a| = -(-a) = a, \]

   by the definition of the absolute value function. Thus,  
   \[ |-a| = |a|, \]

   by the definition of absolute value again.

   Next, suppose that $a < 0$. Then, $-a > 0$, and so, by the definition of the absolute value,  
   \[ |-a| = -a = |a|, \]

   again by the definition of the absolute value.

   Finally, for $a = 0$, we also get $| - a | = |a|$ since $-0 = 0$ and $|0| = 0$.

   We have therefore proved that  
   \[ | - a | = |a| \quad \text{for all} \ a \in \mathbb{R}. \]

\[\square\]
3. Let \( a \) and \( b \) denote real numbers with \( b \neq 0 \). Show that

\[
\left| \frac{a}{b} \right| = \frac{|a|}{|b|}.
\]

**Proof:** Let \( a, b \in \mathbb{R} \) with \( b \neq 0 \). Then, \( b^{-1} \) exists. We first prove that

\[
|b^{-1}| = \frac{1}{|b|}.
\]

To see why this is the case, observe that

\[
b^{-1}b = 1
\]

so that

\[
|b^{-1}b| = 1,
\]

since \( |1| = 1 \), by the definition of absolute value, as \( 1 > 0 \). Thus, by a result proved in class (see Problem 1(b) in Problem Set #2),

\[
|b^{-1}||b| = 1,
\]

from which we get that \( |b| \) is invertible and

\[
|b|^{-1} = |b^{-1}|,
\]

which can be written as

\[
|b^{-1}| = \frac{1}{|b|}.
\]

Next, write

\[
\frac{a}{b} = ab^{-1}.
\]

and take the absolute value of both sides to get

\[
\left| \frac{a}{b} \right| = |a||b^{-1}|,
\]

where we have used again the result of Problem 1(b) in Problem Set #2. Consequently,

\[
\left| \frac{a}{b} \right| = \frac{|a|}{|b|},
\]

which was to be shown. \( \square \)
4. Prove that $|a + b + c| \leq |a| + |b| + |c|$ for all real numbers $a$, $b$ and $c$.

Proof: Apply the triangle inequality to

$$|a + b + c| = |(a + b) + c|$$

to get

$$|a + b + c| \leq |a + b| + |c| \leq |a| + |b| + |c|,$$

where we have used the triangle inequality a second time.

5. Use induction on $n$ to prove that

$$2^n > n \quad \text{for all } n \in \mathbb{N}.$$

Proof: Let $P(n)$ denote the statement “$2^n > n$”. We prove that $P(n)$ is true for all $n \in \mathbb{N}$ by induction on $n$.

First note that $2^1 = 2 = 1 + 1 > 1$, since $1 > 0$. Consequently, $P(1)$ is true.

Next, we prove the implication

$$P(n) \text{ is true } \implies P(n + 1) \text{ is true}.$$

Assume that $P(n)$ is true; that is, $2^n > n$. Consider

$$2^{n+1} = 2 \cdot 2^n = 2^n + 2^n,$$

and apply the assumption that $P(n)$ is true on the right hand side to get

$$2^{n+1} > n + n \geq n + 1,$$

since $n \geq 1$, which shows that $P(n + 1)$ is true.

Hence, by induction on $n$, $2^n > n$ for all $n \in \mathbb{N}$. \qed