Solutions to Assignment #6

1. Let $x \in \mathbb{R}$. Prove that $0 < x \leq 1$ implies that $x^2 \leq x$.

   Proof: Assume that $x > 0$ and $x \leq 1$. Then,
   \[ x \cdot x \leq x \cdot 1, \]
   from which the result follows. \qed

2. Let $a$ and $b$ denote real numbers. Use the triangle inequality to prove that
   \[ ||a| - |b|| \leq |a - b|. \]

   Proof. Write $a = (a - b) + b$ and take absolute value on both sides of this identity to get
   \[ |a| = |(a - b) + b|. \]
   Applying the triangle inequality on the right–hand side we obtain that
   \[ |a| \leq |a - b| + |b|, \]
   from which we get that
   \[ |a| - |b| \leq |a - b|. \] \hspace{1cm} (1)
   Similar calculations show that
   \[ |b| - |a| \leq |b - a|. \]
   Thus, using the fact that $|b - a| = |-(a - b)| = |a - b|$ and multiplying the previous inequality by $-1$, we obtain that
   \[ -|a - b| \leq |a| - |b|. \] \hspace{1cm} (2)
   Combining the inequalities in (1) and (2) yields
   \[ -|a - b| \leq |a| - |b| \leq |a - b|, \]
   which is equivalent to $||a| - |b|| \leq |a - b|$. \qed
3. Let $a$ and $b$ denote positive real numbers. Start with the true statement

$$(a - b)^2 \geq 0$$

to prove the inequality

$$ab \leq \frac{a^2 + b^2}{2}.$$ 

Prove that equality holds if and only if $a = b$.

**Solution:** From the inequality

$$0 \leq (a - b)^2$$

we obtain

$$0 \leq a^2 - 2ab + b^2.$$ 

Adding $2ab$ to both sides of the last inequality we then have that

$$2ab \leq a^2 + b^2,$$

from which the result follows after dividing by 2.

Equality holds if and only if

$$(a - b)^2 = 0,$$

which is true if and only if $a - b = 0$, or $a = b$. □

4. Given a real number $x$, denote by $\max\{x, 0\}$ the larger of $x$ and 0. Prove that

$$\max\{x, 0\} = \frac{x + |x|}{2}.$$ 

**Solution:** We consider two cases: (i) $x \geq 0$, and (ii) $x < 0$.

(i) If $x \geq 0$, then $\max\{x, 0\} = x$. On the other hand

$$\frac{x + |x|}{2} = \frac{x + x}{2} = \frac{2x}{2} = x.$$ 

Thus, the equality is verified in this case.

(ii) If $x < 0$, then $\max\{x, 0\} = 0$, and

$$\frac{x + |x|}{2} = \frac{x - x}{2} = 0.$$ 

So, equality is verified in this case as well.
5. Let $x$ and $\max\{x,0\}$ be as in the previous problem. Denote by $\min\{x,0\}$ the smaller of $x$ and 0. Prove that

$$\min\{x,0\} = -\max\{-x,0\},$$

and use this result to derive a formula for $\min\{x,0\}$ analogous to that for $\max\{x,0\}$ proved in the previous problem.

**Solution:** We consider two cases: (i) $x \geq 0$, and (ii) $x < 0$.

(i) If $x \geq 0$, then $\min\{x,0\} = 0$ and $-x \leq 0$, so that $\max\{-x,0\} = 0$. Thus, equality holds in this case.

(ii) If $x < 0$, then $\min\{x,0\} = x$, and $-x > 0$, so that

$$\max\{-x,0\} = -x.$$ 

Consequently, $-\max\{-x,0\} = x$, which is $\min\{x,0\}$ is this case.

We have therefore established that

$$\min\{x,0\} = -\max\{-x,0\}.$$ 

Using the formula for $\max$ derived in the previous problem we then have that

$$\min\{x,0\} = -\left(\frac{-x + |x|}{2}\right)$$

$$= \frac{x - |x|}{2},$$

since $|x| = |x|$ for all $x \in \mathbb{R}$. Hence,

$$\min\{x,0\} = \frac{x - |x|}{2}.$$ 

□