## Assignment #10

**Due on Wednesday, March 10, 2010**

**Read** Handout #2 on *The Real Numbers System Axioms.*

**Read** Section 4.6 on *Ordered Fields* on pp. 63–66 in Schramm's text.

**Read** Chapter 5 on *Upper Bounds and Suprema,* pp. 80–85, in Schramm's text.

**Read** Section 6.1 on *The Archimedean Property* , pp. 89–91, in Schramm's text.

**Do** the following problems

1. For real numbers $a$ and $b$ with $a < b$, let $(a, b)$ denote the open interval from $a$ to $b$:
$$(a, b) = \{x \in \mathbb{R} \mid a < x < b\}.$$
A subset, $D$, of the real numbers is said to be **dense** in $\mathbb{R}$ if and only if for every open interval, $(a, b)$,
$$(a, b) \cap D \neq \emptyset;$$
that is, the intersection of any open interval with $D$ is nonempty.
Use the fact that between any two distinct real numbers there exists a rational number to prove that $\mathbb{Q}$ is dense in $\mathbb{R}$ according to the definition given above.

2. Show that $\mathbb{Z}$ is not dense in $\mathbb{R}$.

3. Let $a, b \in \mathbb{R}$ with $a < b$. Prove that the set $(a, b) \cap \mathbb{Q}$ is infinite.

4. Given sets $A$ and $B$, the set of elements in $A$ which are not in $B$ is denoted by $A \setminus B$; that is,
$$A \setminus B = \{x \in A \mid x \notin B\}.$$
Thus, for instance, the set $\mathbb{R} \setminus \mathbb{Q}$ is the set of irrational numbers.
Prove that $\mathbb{R} \setminus \mathbb{Q}$ is dense in $\mathbb{R}$.

5. Let $q \in \mathbb{Q}$ and $\alpha$ be an irrational number. Prove that
   (a) if $q \neq 0$, then $q\alpha$ is irrational, and
   (b) $q + \alpha$ is irrational for any $q \in \mathbb{Q}$.
   (c) What can you say about $\alpha^q$?