Assignment #17

Due on Wednesday, April 28, 2010

Do the following problems.

In these problems we see how to define a^x, where a and x and real numbers with $a > 0$. You will need the results that you proved in Problems 4 and 5 of Assignment #16 and Problem 4 in Assignment #14.

1. Let $x \in \mathbb{R}$. Prove that there exists a decreasing sequence, (q_n), of rational numbers which converges to x.

2. Let $x \geqslant 0$ and (q_n) be a sequence of rational numbers which decreased to x. For $a > 1$, define $y_n = a^{q_n}$ for all $n \in \mathbb{N}$. Prove that (y_n) converges by showing that (y_n) is monotone and bounded below.

   Definition. For $x \geqslant 0$ and $a > 1$, we define a^x to be the limit of (a^{q_n}) as $n \to \infty$, where (q_n) is any sequence that decreases to x. By Problem 2, $\lim\limits_{n\to\infty} a^{q_n}$ exists. Thus,

   $$a^x = \lim_{n\to\infty} a^{q_n}.$$

   For this definition to make sense, we must show that if (q_n) and (r_n) are any two sequences of rational numbers that decrease to x, then

   $$\lim_{n\to\infty} a^{q_n} = \lim_{n\to\infty} a^{r_n}.$$

   We will prove this fact in Problems 3 and 4.

3. Let $a > 1$ and (q_n) be a monotone sequence which converges to 0. Prove that

   $$\lim_{n\to\infty} a^{q_n} = 1$$

   Hint: Prove that there is a subsequence, (q_{n_k}), of (q_n) such that

   $$-\frac{1}{k} < q_{n_k} < \frac{1}{k} \quad \text{for all } k \in \mathbb{N}.$$

   Then, use the result of Problem 4 in Assignment #14.

4. Assume that $a > 1$ and let (q_n) and (r_n) be two sequences of rational numbers that decrease to x. Prove that

$$\lim_{n\to\infty} a^{q_n} = \lim_{n\to\infty} a^{r_n}.$$

Hint: Consider $\dfrac{a^{q_n}}{a^{r_n}} = a^{q_n - r_n}$ and use the result of Problem 3.

5. Let $a > 0$ and $x \in \mathbb{R}$.

   (a) Explain how to define a^x.

   (b) For real numbers, x and y, prove that $a^{x+y} = a^x a^y$.

   (c) For real numbers, x and y, prove that $a^{x-y} = \dfrac{a^x}{a^y}$.