Assignment #17

Due on Wednesday, April 28, 2010

Do the following problems.

In these problems we see how to define \( a^x \), where \( a \) and \( x \) are real numbers with \( a > 0 \). You will need the results that you proved in Problems 4 and 5 of Assignment #16 and Problem 4 in Assignment #14.

1. Let \( x \in \mathbb{R} \). Prove that there exists a decreasing sequence, \((q_n)\), of rational numbers which converges to \( x \).

2. Let \( x \geq 0 \) and \((q_n)\) be a sequence of rational numbers which decreased to \( x \). For \( a > 1 \), define \( y_n = a^{q_n} \) for all \( n \in \mathbb{N} \). Prove that \((y_n)\) converges by showing that \((y_n)\) is monotone and bounded below.

**Definition.** For \( x \geq 0 \) and \( a > 1 \), we define \( a^x \) to be the limit of \((a^{q_n})\) as \( n \to \infty \), where \((q_n)\) is any sequence that decreases to \( x \). By Problem 2, \( \lim_{n \to \infty} a^{q_n} \) exists. Thus,

\[
a^x = \lim_{n \to \infty} a^{q_n}.
\]

For this definition to make sense, we must show that if \((q_n)\) and \((r_n)\) are any two sequences of rational numbers that decrease to \( x \), then

\[
\lim_{n \to \infty} a^{q_n} = \lim_{n \to \infty} a^{r_n}.
\]

We will prove this fact in Problems 3 and 4.

3. Let \( a > 1 \) and \((q_n)\) be a monotone sequence which converges to \( 0 \). Prove that

\[
\lim_{n \to \infty} a^{q_n} = 1
\]

**Hint:** Prove that there is a subsequence, \((q_{n_k})\), of \((q_n)\) such that

\[
-\frac{1}{k} < q_{n_k} < \frac{1}{k} \quad \text{for all } k \in \mathbb{N}.
\]

Then, use the result of Problem 4 in Assignment #14.
4. Assume that \( a > 1 \) and let \( (q_n) \) and \( (r_n) \) be two sequences of rational numbers that decrease to \( x \). Prove that

\[
\lim_{n \to \infty} a^{q_n} = \lim_{n \to \infty} a^{r_n}.
\]

*Hint:* Consider \( \frac{a^{q_n}}{a^{r_n}} = a^{q_n-r_n} \) and use the result of Problem 3.

5. Let \( a > 0 \) and \( x \in \mathbb{R} \).

(a) Explain how to define \( a^x \).

(b) For real numbers, \( x \) and \( y \), prove that \( a^{x+y} = a^x a^y \).

(c) For real numbers, \( x \) and \( y \), prove that \( a^{x-y} = \frac{a^x}{a^y} \).