Solutions to Exam 1 (Part I)

1. Provide concise answers to the following questions:

(a) A subset, $A$, of the real numbers is said to be **bounded** if there exists a positive real number, $M$, such that

$$|a| \leq M \quad \text{for all } a \in A.$$

Give the negation of the statement

“$A$ is bounded.”

**Answer:** The negation of "$A$ is bounded" is

For every positive number, $M$, there exists an element, $a$, in $A$ such that $|a| > M$.

(b) Let $A$ denote a subset of the real numbers and $\beta$ a positive real number. Give the contrapositive for the following implication:

$$t \in A \Rightarrow t \leq s - \beta.$$

**Answer:** The contrapositive of “$t \in A \Rightarrow t \leq s - \beta$” is

$$t > s - \beta \Rightarrow t \notin A.$$

2. Use the field and order axioms of the real numbers to prove the following.

(a) Let $a, b \in R$. If $ab = 0$, then either $a = 0$ or $b = 0$.

**Proof:** Assume that $ab = 0$ and $a \neq 0$. Then, by Field Axiom $(F_9)$, $a^{-1}$ exists. Multiplying

$$ab = 0$$

by $a^{-1}$ on both sides yields

$$a^{-1}(ab) = a^{-1}\cdot 0 = 0,$$

from which we get that $b = 0$, where we have used the Field Axioms $(F_7)$, $(F_9)$ and $(F_{10})$. □
(b) Let \( p \in \mathbb{R} \). If \( p > 1 \), then \( p < p^2 \).

Proof: Assume that \( p > 1 \). It then follows that \( p > 0 \), since \( 1 > 0 \). We also have that \( p - 1 > 0 \). Consequently, by the Order Axiom \((O_3)\),

\[
p(p - 1) > 0.
\]

Thus, by the distributive property,

\[
p^2 - p > 0,
\]

from which we get that \( p < p^2 \). \( \square \)

3. Use the completeness axiom of \( \mathbb{R} \) to prove that the set of natural numbers is not bounded above. Deduce, therefore, that for any real number, \( x \), there exists a natural number, \( n \), such that

\[
x < n.
\]

Proof: Assume by way of contradiction that \( \mathbb{N} \) is a bounded above. Then, since \( \mathbb{N} \) is not empty, it follows from the completeness axiom that \( \text{sup}(\mathbb{N}) \) exists. Thus there must be \( m \in \mathbb{N} \) such that

\[
\text{sup}(\mathbb{N}) - 1 < m. \tag{1}
\]

It follows from the inequality in (1) that

\[
\text{sup}(\mathbb{N}) < m + 1,
\]

where \( m + 1 \in \mathbb{N} \). This is a contradiction. Therefore, it must be that case that \( \mathbb{N} \) not bounded above.

Thus, given any real number, \( x \), there must be a natural number, \( n \), such that

\[
x < n.
\]

Otherwise,

\[
m \leq x \quad \text{for all } m \in \mathbb{N},
\]

which would say that \( x \) is an upper bound for \( \mathbb{N} \). But we just proved that \( \mathbb{N} \) is not bounded above. \( \square \)