Solutions to Exam 2 (Part II)

1. Let (x_n) denote a sequence of real numbers. For a fixed $N_o \in \mathbb{N}$, define

$$y_n = x_{_{N_o+n}} \quad \text{for all } n \in \mathbb{N};$$

that is; $y_1 = x_{_{N_o+1}}$, the $(N_o+1)^{\text{th}}$ term in the sequence (x_n), y_2 is the $(N_o+2)^{\text{th}}$ term, and so on.

(a) Prove that (x_n) converges if and only if (y_n) converges.

Proof: Suppose that (x_n) converges to $x \in \mathbb{R}$. We show that (y_n) also converges to x.
Let $\varepsilon > 0$ be given. Then there exists $N_1 \in \mathbb{N}$ such that

$$n \geqslant N_1 \Rightarrow |x_n - x| < \varepsilon. \tag{1}$$

We may choose $N_1 > N_o$. Then $N_1 - N_o \in \mathbb{N}$. Let $N = N_1 - N_o$. Then, $n \geqslant N$ implies that $N_o + N \geqslant N_1$, so that, by virtue of (1),

$$|y_n - x| = |x_{_{N_o+n}} - x| < \varepsilon.$$

Thus, we have shown that

$$\lim_{n\to\infty} y_n = x.$$

Conversely, assume that (y_n) converges to $y \in \mathbb{R}$. We show that (x_n) also converges to y.
Let $\varepsilon > 0$ be given. Then there exists $N_2 \in \mathbb{N}$ such that

$$n \geqslant N_2 \Rightarrow |y_n - y| < \varepsilon. \tag{2}$$

Let $N = N_2 + N_o$. Then, $n \geqslant N$ implies that $n - N_o \geqslant N_2$, so that, by virtue of (2),

$$|x_n - y| = |x_{_{n-N_o+N_o}} - y| = |y_{_{n-N_o}} - y| < \varepsilon.$$

Thus, (y_n) converges to y implies that

$$\lim_{n\to\infty} x_n = y.$$

$\square$

(b) Prove that if (x_n) is bounded and (y_n) is monotone, the both (x_n) and (y_n) converge.

Proof: Assume that (x_n) is bounded and (y_n) is monotone. Then, there exists $M > 0$ such that

$$|x_n| \leqslant M \quad \text{for all } n \in \mathbb{N}.$$

It then follows that

$$|y_n| = |x_{N_o+n}| \leqslant M \quad \text{for all } n \in \mathbb{N};$$

that is, the sequence (y_n) is bounded. Since (y_n) is also monotone, (y_n) converges by the Bounded, Monotone Convergence Theorem. Hence, by the result of part (a), (x_n) also converges. □

(c) Give an interpretation of the results in this problem.

Answer: The convergence properties of a sequence, (x_n), are completely determined by the terms of the sequence after some $N_o \in \mathbb{N}$; in other words, by the properties of the sequence

$$(x_{N_o+1}, x_{N_o+2}, x_{N_o+3}, \ldots).$$

□

2. Define a sequence, (x_n), of real numbers as follows:

$$\begin{aligned} x_1 &= 1; \\ x_{n+1} &= \sqrt{1+x_n} \quad \text{for all } n \in \mathbb{N}. \end{aligned}$$

(a) Prove that (x_n) is monotone.

Suggestion: Consider $x_{n+2}^2 - x_{n+1}^2$

Proof: We show that $x_{n+1} > x_n$ for all $n \in \mathbb{N}$ by induction on n. For $n = 1$, note that $x_{1+1} = \sqrt{1+x_1} = \sqrt{2} > 1 = x_1$. So, the result is true for $n = 1$. Next, assume that

$$x_{n+1} > x_n, \tag{3}$$

and consider

$$\begin{aligned} x_{n+2}^2 - x_{n+1}^2 &= 1 + x_{n+1} - (1 + x_n) \\ &= x_{n+1} - x_n. \end{aligned}$$

It then follows from the inductive hypothesis in (3) that

$$x_{n+2}^2 - x_{n+1}^2 > 0,$$

from which we get that

$$x_{n+2} > x_{n+1}.$$

It then follows that the sequence (x_n) is increasing. □

(b) Show that $x_n < 2$ for all $n \in \mathbb{N}$.

Proof: We argue by induction on n. Note that $x_1 = 1 < 2$; so the result is true for $n = 1$.

Next, assume that

$$x_n < 2. \tag{4}$$

We then have that

$$x_{n+1} = \sqrt{1 + x_n} < \sqrt{1 + 2},$$

by the inductive hypothesis in (4). Thus,

$$x_{n+1} < \sqrt{3} < 2,$$

since $3 < 4$. The inductive argument is now complete. □

(c) Deduce that (x_n) converges.

Solution: By parts (a) and (b), the sequence (x_n) is monotone and bounded. Hence, by the Bounded, Monotone Convergence Theorem, (x_n) converges. □

(d) Compute the limit of (x_n).

Solution: Let x denote the limit of the sequence (x_n). Then, by the result of part (a) in Problem 1, with $N_o = 1$,

$$\lim_{n\to\infty} x_{n+1} = x,$$

from which we get that

$$\lim_{n\to\infty} x_{n+1}^2 = x^2.$$

Thus, taking the limit as $n \to \infty$ on both sides of

$$x_{n+1}^2 = 1 + x_n$$

yields that

$$x^2 = 1 + x.$$

Hence, x is the positive solution of the quadratic equation

$$x^2 - x - 1 = 0,$$

or

$$x = \frac{1 + \sqrt{5}}{2}.$$

□