Solutions to Review Problems for Exam #1

1. Let $B$ denote a non-empty subset of the real numbers which is bounded below. Define

$$A = \{x \in \mathbb{R} \mid x \text{ is a lower bound for } B\}.$$ 

Prove that $A$ is non-empty and bounded above, and that $\sup A = \inf B$.

**Solution:** Since $B$ is bounded below, there exists $\ell \in \mathbb{R}$ such that $\ell$ is a lower bound for $B$. Hence, $\ell \in A$ and, therefore, $A$ is not empty. Next, use the assumption that $B$ is non-empty to conclude that there exists $b \in B$. Then, for any lower bound, $\ell$, of $B$,

$$\ell \leq b.$$ 

Hence, $b$ is an upper bound for $A$.

Thus, we have shown that $A$ is non-empty and bounded above. Therefore, by the Completeness Axiom, $\sup(A)$ exists.

We show next that $\sup(A)$ is the infimum of $B$.

First we show that $\sup(A)$ is a lower bound for $B$. Let $\ell \in A$, then

$$\ell \leq b \quad \text{for every } b \in B.$$ 

Thus, every $b \in B$ is an upper bound for $A$. Consequently,

$$\sup(A) \leq b \quad \text{for every } b \in B.$$ 

Hence, $\sup(A)$ is a lower bound for $B$.

Next, let $c$ be a lower bound for $B$. Then $c \in A$ and therefore

$$c \leq \sup(A);$$ 

that is, $\sup(A)$ is greater or equal to any lower bound for $B$. In other words,

$$\sup(A) = \inf(B),$$ 

which was to be shown. \qed

2. Prove that, for any real number, $x$,

$$|x^2| = |x|^2 = x^2.$$
Proof: Compute

\[ |x^2| = |xx| \]
\[ = |x||x| \]
\[ = |x|^2. \]

On the other hand, by the definition of the absolute value function,

\[ |x^2| = x^2, \]

since \( x^2 \geq 0 \). It then follows that \( |x|^2 = x^2 \), and the proof is now complete. \( \square \)

3. Let \( a, b, c \in \mathbb{R} \) with \( c > 0 \). Show that \( |a - b| < c \) if and only if \( b - c < a < b + c \).

Solution: \( |a - b| < c \) if and only if \( -c < a - b < c \), which is true if and only if

\[ b - c < a < b + c, \]

where we have added \( b \) to each part of the inequality. \( \square \)

4. Let \( a, b \in \mathbb{R} \). Show that if \( a < x \) for all \( x > b \), then \( a \leq b \).

Proof: Assume, by way of contradiction, that \( a < x \) for all \( x > b \) and \( a > b \). It then follows that \( a < a \), which is absurd. Hence, \( a < x \) for all \( x > b \) implies that \( a \leq b \). \( \square \)

5. Show that the set \( A = \{1/n \mid n \in \mathbb{N}\} \) is bounded above and below, and give its supremum and infimum.

Solution: Observe that \( \frac{1}{n} \leq 1 \) for all \( n \in \mathbb{N} \). It then follows that 1 is an upper bound for \( A \). Since, \( A \neq \emptyset \), \( \text{sup}(A) \) exists and

\[ \text{sup}(A) \leq 1. \]

To see that \( \text{sup}(A) = 1 \), observe that 1 \( \in A \) and therefore \( 1 \leq \text{sup}(A) \).
Next, observe that \( n > 0 \) for all \( n \in \mathbb{N} \). It then follows that \( n^{-1} > 0 \) for all \( n \in \mathbb{N} \). Thus, 0 is a lower bound for \( A \). Consequently, the infimum of \( A \) exists and
\[
0 \leq \inf(A).
\]
To see that \( \inf(A) = 0 \), assume to the contrary that \( \inf(A) > 0 \); then
\[
\frac{1}{\inf(A)} > 0.
\]
Since \( \mathbb{N} \) is unbounded, there exists a natural number, \( n \), such that
\[
n > \frac{1}{\inf(A)}.
\]
It then follows that
\[
\frac{1}{n} < \inf(A),
\]
which is impossible since \( \frac{1}{n} \in A \). Thus, \( \inf(A) = 0 \). \( \square \)

6. Let \( A = \{ n + \frac{(-1)^n}{n} \mid n \in \mathbb{N} \} \). Compute \( \sup A \) and \( \inf A \), if they exist.

**Solution:** First note that, since
\[
\left| \frac{(-1)^n}{n} \right| = \frac{1}{n} \leq 1,
\]
for all \( n \in \mathbb{N} \), it follows that
\[
n + \frac{(-1)^n}{n} \geq n - \left| \frac{(-1)^n}{n} \right| \geq n - 1 \tag{1}
\]
for all \( n \in \mathbb{N} \). Consequently, the set \( A \) is not bounded since \( \mathbb{N} \) is unbounded. Therefore, \( \sup(A) \) does not exist.

On the other hand, it follows from the inequality in (1) that
\[
n + \frac{(-1)^n}{n} \geq 0
\]
for all \( n \in \mathbb{N} \). Thus, 0 is a lower bound for \( A \). Therefore, since \( A \) is not empty, \( \inf(A) \) exists and
\[
\inf(A) \geq 0.
\]
To see that \( \inf(A) = 0 \), note that \( 0 \in A \). \( \square \)
7. Let \( A = \{1/n \mid n \in \mathbb{N} \text{ and } n \text{ is prime}\} \). Compute \( \sup A \) and \( \inf A \), if they exist.

**Solution:** Since \( n = 2 \) is the smallest prime, it follows that \( n \geq 2 \) for all \( n \in \mathbb{N} \) which are prime. It then follows that 
\[
a \leq \frac{1}{2} \quad \text{for all } a \in A.
\]
Thus, \( \frac{1}{2} \) is an upper bound for \( A \). Hence, since \( A \) is non-empty, \( \sup(A) \) exists and 
\[
\sup(A) \leq \frac{1}{2}.
\]
In fact, \( \sup(A) = \frac{1}{2} \) since \( \frac{1}{2} \in A \).

Next, note that, by definition, prime numbers are positive. Consequently, \( a > 0 \) for all \( a \in A \) and therefore 0 is a lower bound for \( A \). Thus, \( \inf(A) \) exists and 
\[
\inf(A) \geq 0.
\]
To see that \( \inf(A) = 0 \), argue by contradiction. If \( \inf(A) > 0 \), then 
\[
\frac{1}{\inf(A)} > 0,
\]
and so, since the set of primes is unbounded, there exists a prime number, \( p \), with 
\[
\frac{1}{\inf(A)} < p,
\]
from which we get that 
\[
\inf(A) > \frac{1}{p},
\]
which is impossible since \( \frac{1}{p} \in A \). Therefore, \( \inf(A) = 0. \) \( \square \)

8. Let \( A \) denote a subset of \( \mathbb{R} \). Give the negation of the statement: “\( A \) is bounded above.”

**Solution:** First, translate the statement “\( A \) is bounded above” into 
\[
\exists u \in \mathbb{R} \text{ such that } (\forall a \in A) \ a \leq u.
\]
Thus, the negation of the statement reads 
\[
(\forall u \in \mathbb{R}) \ (\exists a \in A) \text{ such that } a > u.
\]
In other words, “for every real number, \( u \), it is possible to find an element of \( A \) which is bigger than \( u \).” \( \square \)

9. Let \( A \subseteq \mathbb{R} \) be non–empty and bounded from above. Put \( s = \sup A \). Prove that for every \( n \in \mathbb{N} \) there exists \( x_n \in A \) such that

\[
s - \frac{1}{n} < x_n \leq s.
\]

**Proof:** Note that for all \( n \in \mathbb{N} \), \( \frac{1}{n} > 0 \). Thus,

\[
s - \frac{1}{n} < s.
\]

Thus, for each \( n \in \mathbb{N} \), it is possible to find an element of \( A \), call it \( x_n \), such that

\[
s - \frac{1}{n} < x_n;
\]

otherwise,

\[
x \leq s - \frac{1}{n} \quad \text{for all } x \in A,
\]

which would say that \( s - \frac{1}{n} \) is an upper bound of \( A \), smaller than \( \sup(A) \). This is impossible. Hence, for every \( n \in \mathbb{N} \) there exists \( x_n \in A \) such that

\[
s - \frac{1}{n} < x_n \leq s.
\]

\( \square \)

10. What can you say about a non–empty subset, \( A \), of real numbers for which \( \sup A = \inf A \).

**Solution:** Assume that \( A \subseteq \mathbb{R} \) is non–empty with \( \sup(A) = \inf(A) \).

Let \( a \) denote any element in \( A \). Then,

\[
\sup(A) = \inf(A) \leq a \leq \sup(A),
\]

which shows that \( a = \sup(A) \). Thus,

\[
A = \{\sup(A)\};
\]

in other words, \( A \) consists of a single element, \( \sup(A) \). \( \square \)