Math 101. Rumbos

Review Problems for Exam #2

1. Suppose that the sequence \((x_n)\) converges to \(a \neq 0\), where \(x_n \neq 0\) for all \(n \in \mathbb{N}\).
   Prove that the sequence \(\left(\frac{1}{x_n}\right)\) converges to \(\frac{1}{a}\).

2. Let \((x_n)\) denote a sequence that converges to \(x\). Prove that for any \(m \in \mathbb{N}\),
   \[ \lim_{n \to \infty} x_n^m = x^m. \]

3. Let \(\delta > 0\) and define \(y_n = \frac{1}{(1 + \delta)^n}\) for all \(n \in \mathbb{N}\).
   (a) Use the estimate \((1 + \delta)^n > n\delta\), for all \(n \in \mathbb{N}\), to prove that the sequence
       \((y_n)\) converges to 0.
   (b) Define \(x_n = x^n\). Prove that if \(|x| < 1\), then \((x_n)\) converges. What is
       \( \lim_{n \to \infty} x_n? \)

4. Let \((x_n)\) denote a sequence of real numbers.
   (a) Prove that if \((x_n)\) converges then \((x_n^2)\) converges.
   (b) Show that the converse of the statement in part (a) is not true.

5. Let \(x\), \(a\) and \(b\) denote a real numbers.
   (a) Derive the factorization:
       \[ x^n - 1 = (x - 1)(x^{n-1} + x^{n-2} + \cdots + x + 1). \]
       \(Suggestion:\) Let \(S = 1 + x + x^2 + \cdots + x^{n-2} + x^{n-1}\) and compute \(xS\) and \(xS - S\).
   (b) Derive the factorization formula
       \[ a^n - b^n = (a - b)(a^{n-1} + a^{n-2}b + \cdots + b^{n-1}) \]
   (c) Let \(a\) and \(b\) denote positive real numbers, and \(n\) a natural number. Prove that
       \(a > b\) if and only if \(a^n > b^n\).
6. Given \( a > 0 \) and \( n \in \mathbb{N} \), prove that there exists a unique positive solution to the equation \( x^n = a \).

\textit{Note:} In this problem, you might need to use the binomial expansion
\[
(a + b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}, \quad \text{where} \quad \binom{n}{k} = \frac{n!}{k!(n-k)!}, \quad \text{for} \quad k = 0, 1, 2, \ldots, n.
\]

7. Let \( a \) and \( b \) denote positive real numbers. For each natural number \( n \), let \( a^{1/n} \) denote the unique positive solution to the equation \( x^n = a \).

(a) Prove that if \( b \leq 1 \), then \( b^m \leq 1 \) for all \( m \in \mathbb{N} \).
(b) Show that if \( a > 1 \), then \( a^{1/n} > 1 \) for all \( n \in \mathbb{N} \).
(c) Prove that if \( a > 1 \), then \( a^{m/n} > 1 \) for all \( m, n \in \mathbb{N} \), where \( a^{m/n} = (a^{1/n})^m \).

8. Let \( a \) and \( b \) denote positive real, and \( n \) a natural number. Prove that
\[ a > b \quad \text{if and only if} \quad a^{1/n} > b^{1/n}. \]

9. Let \( a \) denote a positive real number.

(a) Show that if \( a > 1 \), then \( a - 1 > n(a^{1/n} - 1) \) for all \( n \in \mathbb{N} \). Deduce that \( \lim_{n \to \infty} a^{1/n} = 1 \), for \( a > 1 \).
(b) Prove that for any positive real number \( a \), \( \lim_{n \to \infty} a^{1/n} = 1 \).

10. Define \( x_n = 1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots + \frac{1}{2^{n-1}} \) for \( n \in \mathbb{N} \).

(a) Multiply the expression for \( x_n \) by \( 1/2 \) and obtain that \( x_n = 2 - \frac{2}{2^n} \) for \( n \in \mathbb{N} \).
(b) Deduce that \( (x_n) \) converges to 2.

11. Define \( s_n = \sum_{k=0}^{n} \frac{1}{k!} = 1 + 1 + \frac{1}{2} + \frac{1}{3!} + \frac{1}{4!} + \cdots + \frac{1}{n!} \) for all \( n = 1, 2, 3, \ldots \)

(a) Show that \( s_n \leq 1 + x_n \), where \( x_n = 1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots + \frac{1}{2^{n-1}} \), for all \( n = 1, 2, 3, \ldots \).
(b) Show that the sequence \( (s_n) \) is increasing and bounded and, therefore, it converges.
(c) Denote the limit of \( (s_n) \) by \( e \) and show that \( 2.5 \leq e \leq 3 \).