I. Field Axioms

The set of real numbers \( \mathbb{R} \) has two algebraic operations: \textit{addition} (the sum of any two elements \( x \) and \( y \) of \( \mathbb{R} \) being denoted by \( x + y \)) and \textit{multiplication} (the product of any two elements \( x \) and \( y \) of \( \mathbb{R} \) being denoted by \( xy \)) defined for any pair of elements in the set. These operations satisfy the properties of a \textit{field}, which are the following:

Closure properties

\((F_1)\) For any two real numbers \( x \) and \( y \), \( x + y \) and \( xy \) are real numbers.

Properties of addition

\((F_2)\) (Commutativity). For any \( x \) and \( y \) in \( \mathbb{R} \),
\[ x + y = y + x. \]

\((F_3)\) (Associativity). For any three elements \( x, y, \) and \( z \) in \( \mathbb{R} \),
\[ (x + y) + z = x + (y + z). \]

\((F_4)\) (Existence of an additive identity). There exists a real number 0 with the property:
\[ x + 0 = x \quad \text{for all } x \text{ in } \mathbb{R}. \]

\((F_5)\) (Existence of additive inverses). For every \( x \) in \( \mathbb{R} \), there exists \( y \) in \( \mathbb{R} \) with the property:
\[ x + y = 0. \]

Properties of multiplication

\((F_6)\) (Commutativity). For any pair of real numbers \( x \) and \( y \),
\[ xy = yx. \]

\((F_7)\) (Associativity). For any three elements \( x, y, \) and \( z \) in \( \mathbb{R} \),
\[ (xy)z = x(yz). \]
(F₈) (*Existence of an multiplicative identity*). There exists a real number 1 such that \(1 \neq 0\) and 
\[ x \cdot 1 = x \quad \text{for all } x \in \mathbb{R}. \]

(F₉) (*Existence of multiplicative inverses for non–zero real numbers*). For every \(x\) in \(\mathbb{R}\) such that \(x \neq 0\), there exists \(y\) in \(\mathbb{R}\) such that 
\[ xy = 1. \]

**Distributive property**

(F₁₀) For any real numbers \(x, y\) and \(z\), 
\[ x(y + z) = xy + xz. \]

### II. Order Axioms

We designate a certain subset \(P\) of \(\mathbb{R}\) as the “positive numbers” in \(\mathbb{R}\). This set \(P\) is “invariant” under the operations in \(\mathbb{R}\); i.e., if \(x\) and \(y\) are in \(P\), then \(x + y\) and \(xy\) are also in \(P\). The set \(P\) induces an **order relation** in \(\mathbb{R}\) as follows: we say that \(x < y\) if \(y - x \in P\). The notation \(x \leq y\) means \(x < y\) or \(x = y\). Similarly, we define \(x > y\) to mean \(x - y \in P\), and \(x \geq y\) to mean \(x > y\) or \(x = y\).

The field \(\mathbb{R}\) is an **ordered field** since the following properties hold:

(O₁) (*Trichotomy property*). If \(x \in \mathbb{R}\), then \(x = 0\) or \(x > 0\) or \(x < 0\). (Note: only one of these three possibilities can hold.)

(O₂) If \(x > 0\) and \(y > 0\), then \(x + y > 0\).

(O₃) If \(x > 0\) and \(y > 0\), then \(xy > 0\).

### III. Completeness Axiom

Let \(A\) be a subset of \(\mathbb{R}\). We say that \(b\) is an **upper bound** for \(A\) if \(x \leq b\) for all \(x \in A\). A number \(c\) is called a **least upper bound** for \(A\) if \(c\) is an upper bound for \(A\) and \(c \leq b\) for any upper bound \(b\) for \(A\). The ordered field \(\mathbb{R}\) is said to be **complete** since it satisfies the following

(C) (*Least upper bound property*). Every non-empty subset of \(\mathbb{R}\) that has an upper bound has a least upper bound.
Remarks

1. Given \( x \in \mathbb{R} \), the additive inverse for \( x \) given by the field axiom \((F_5)\) is unique and is denoted by \(-x\). The expression \( y - x \), for any pair of real numbers \( x \) and \( y \), is then interpreted as \( y + (-x) \).

2. Given a non–zero real number \( x \), the multiplicative inverse for \( x \) given by the field axiom \((F_9)\) is unique and is denoted by \( x^{-1} \) or \( \frac{1}{x} \). The expression \( \frac{y}{x} \), for \( x, y \in \mathbb{R} \) with \( x \neq 0 \), is then interpreted as \( y x^{-1} \) or \( \frac{y}{x} \).

3. The set of rational numbers \( \mathbb{Q} \) is a sub-field of \( \mathbb{R} \); that is, the field axioms \((F_1)–(F_{10})\) hold true for \( \mathbb{Q} \) as well. The rational numbers are also an ordered field with the same order relation defined in \( \mathbb{R} \). However, \( \mathbb{Q} \) is not a complete field.