Problem Set #1: The Set of Real Numbers

1. Let \( n \) be a natural number. Show that if \( n \) is even, then so is \( n^2 \). Conversely, show that if \( n^2 \) is even, then \( n \) is even.

2. Let \( n \) be a natural number. Show that \( n \) is a multiple of 3 if and only if \( n^2 \) is a multiple of 3.

3. Show that \( \sqrt{3} \) is irrational.

4. Show that \( \sqrt{6} \) is irrational.

5. True or false.
   
   (a) If \( \alpha \) and \( \beta \) are irrational, then \( \alpha + \beta \) is irrational.
   
   (b) If \( \alpha \) and \( \beta \) are irrational, then \( \alpha \beta \) is irrational.

6. The following are consequences of the field axioms for the real numbers:

   (a) (The cancelation laws)
      
      i. Let \( x, y \) and \( z \) be real numbers. If \( x + z = y + z \), then \( x = y \).
      
      ii. Let \( x, y \) and \( z \) be real numbers with \( z \neq 0 \). If \( xz = yz \), then \( x = y \).

   (b) Show that 0 and 1 are unique.

   (c) Given \( x \in \mathbb{R} \), \( -x \) and \( x^{-1} \) are unique.

   (d) \( x \cdot 0 = 0 \) for all \( x \in \mathbb{R} \).

   (e) \( (-1) \cdot (-x) = x \) for all \( x \in \mathbb{R} \), where \( -x \) is the unique additive inverse of \( x \) given by the field axiom (\( F_5 \)).

   (f) Let \( a, b \) and \( c \) be real numbers with \( a \neq 0 \), then the equation \( ax + b = c \) has a unique solution.

7. Let \( a \) and \( b \) be real numbers. If \( ab = 0 \), then either \( a = 0 \) or \( b = 0 \).

8. Show that the set of rational numbers \( \mathbb{Q} \) is a sub–field of the set of real numbers.
9. The following are consequences of the field and order axioms for the real numbers. Let $x$, $y$ and $z$ be real numbers.

(a) If $x < y$ and $y < z$, then $x < z$.
(b) If $x < y$, then $x + z < y + z$.
(c) If $x < y$ and $z > 0$, then $xz < yz$.
(d) If $x < 0$ and $y < 0$, then $xy > 0$.
(e) If $x < y$ and $z < 0$, then $yz < xz$.
(f) If $x < y$, then $-y < -x$.
(g) $x^2 \geq 0$ for any real number $x$.
(h) $1 > 0$
(i) If $x > 0$, then $x^{-1} > 0$.
(j) If $0 < x < y$, then $0 < \frac{1}{y} < \frac{1}{x}$.

10. For any $x \in \mathbb{R}$, $x \leq |x|$.

11. Show that $\mathbb{Q}$ is an ordered subfield of $\mathbb{R}$. 