Problem Set #2: Inequalities

1. Given any real number x, we define the **absolute value** of x to be

$$|x| = \begin{cases} x & \text{if } x \geqslant 0, \\ -x & \text{if } x < 0. \end{cases}$$

   Prove the following statements:

   (a) For any $x \in \mathbb{R}$, $|x| \geq 0$, and $|x| = 0$ iff $x = 0$.

   (b) For any real numbers a and b, $|ab| = |a||b|$.

   (c) For any real numbers a and b with $b > 0$, $|a| < b$ if and only if $-b < a < b$.

   (d) For any real numbers a and b with $b > 0$, $|a| > b$ if and only if $a < -b$ or $a > b$.

   (e) For any real number x, $|x|^2 = x^2$. Conclude therefore that $|x| = \sqrt{x^2}$.

2. Let x and y be real numbers such that $x > 0$ and $y > 0$.

   (a) Prove that $x < y$ iff $x^2 < y^2$.

   (b) Prove that $x < y$ iff $\sqrt{x} < \sqrt{y}$.

3. Let a and b be real numbers.

   (a) *(The Triangle Inequality).* Prove that $|a + b| \leqslant |a| + |b|$.

   (b) Prove that $||a| - |b|| \leq |a - b|$.

4. Let a and b be **positive** real numbers. Prove the following inequalities.

   (a) $\sqrt{ab} \leq \dfrac{a+b}{2}$. Equality holds iff $a = b$.

   (b) $\sqrt{a^2 + b^2} \leq a + b$.

5. Let a be a real number satisfying $|a| < \varepsilon$ for every $\varepsilon > 0$. Prove that $a = 0$.

6. Let a and b be a real numbers satisfying $a < b + \varepsilon$ for every $\varepsilon > 0$. Prove that $a \leq b$.

7. Let x be any real number. Show that

   (a) $\max\{x, 0\} = \dfrac{x + |x|}{2}$, and

   (b) $\min\{x, 0\} = \dfrac{x - |x|}{2}$.