Problem Set #3: Completeness Axiom (Part I)

Read: Chapter 5 on Upper Bounds and Suprema, pp. 80–85, in Michael J. Schramm’ book: “Introduction to Real Analysis.”

Definitions and Notation.
Given a non–empty subset $A$ of $\mathbb{R}$, if $A$ is bounded from above, then the least upper bound of $A$ exists by the Completeness Axiom. We shall denote the least upper bound by $\text{sup } A$ and refer to it also as the supremum of the set $A$.

Let $B$ be a non–empty subset of $\mathbb{R}$. We say that $B$ is bounded from below if there exists $s \in \mathbb{R}$ such that $s \leq x$ for all $x \in B$. We call $s$ a lower bound for $B$. A lower bound $b$ for $B$ with the property that $s \leq b$ for any lower bound $s$ of $B$ is called the greatest lower bound for $B$. If it exists, we denote the greatest lower bound of $B$ by $\text{inf } B$ and refer to it also as the infimum of $B$.

Problems:

1. Let $A$ be a non–empty subset of $\mathbb{R}$. Prove that if $\text{inf } A$ and $\text{sup } A$ exist, then they are unique.

2. Let $B$ be a non–empty subset of $\mathbb{R}$. Denote by $-B$ the set $\{x \in \mathbb{R} \mid -x \in B\}$.

   (a) Prove that if $B$ is bounded from below, then $-B$ is bounded from above. Consequently, $\text{sup } (-B)$ exists.

   (b) Prove that if $B$ is bounded from below, then $B$ has a greatest lower bound and that

\[ \text{inf } B = - \text{sup } (-B). \]

3. Let $A$ and $B$ be subsets of $\mathbb{R}$ and define $A + B$ to be the set

\[ \{z \in \mathbb{R} \mid z = x + y, \text{ where } x \in A \text{ and } y \in B\}. \]

   (a) Prove that if $A$ and $B$ are non–empty and bounded from above, then so is $A + B$ and that

\[ \text{sup } (A + B) \leq \text{sup } A + \text{sup } B. \]

   (b) Prove that if $A$ and $B$ are non–empty and bounded from below, then so is $A + B$ and that

\[ \text{inf } (A + B) \geq \text{inf } A + \text{inf } B. \]
4. Let $A$ and $B$ be non–empty subsets of $\mathbb{R}$. Prove the following statements.

(a) If $A \subseteq B$ and $B$ is bounded from above, then $\sup A \leq \sup B$.

(b) If $A \subseteq B$ and $B$ is bounded from below, then $\inf B \leq \inf A$.

5. For a subset $A$ of the real numbers and $c \in \mathbb{R}$, define: (i) $A + c = \{y \in \mathbb{R} \mid y = x + c \text{ where } x \in A\}$, and (ii) $cA = \{y \in \mathbb{R} \mid y = cx \text{ where } x \in A\}$. Prove the following statements.

(a) Suppose that $A$ is non–empty and bounded above, then
$$\sup(A + c) = \sup A + c.$$ 

(b) Suppose that $A$ is non–empty and bounded above. If $c > 0$, then
$$\sup(cA) = c \sup A.$$ 

(c) Suppose that $A$ is non–empty and bounded above. If $c < 0$, then
$$\inf(cA) = c \sup A.$$ 

6. Let $A$ be a non–empty subset of $\mathbb{R}$ which is bounded from above. Then, $s = \sup A$ if and only if

(i) for each $\varepsilon > 0$, if $x \in A$ then $x < s + \varepsilon$, and

(ii) for each $\varepsilon > 0$ there exists $x \in A$ such that $s - \varepsilon < x$.

7. State and prove a result analogous to that of problem (7) for a non–empty subset $B$ of $\mathbb{R}$ which is bounded from below.

8. Let $A$ be a non–empty subset of $\mathbb{R}$. Prove that if $u$ is an upper bound for $A$ and $u \in A$, then $u = \sup A$.

9. Let $A$ and $B$ be a non–empty subsets of $\mathbb{R}$ which are bounded from above. Prove that if $\sup A < \sup B$, then there exists $b \in B$ such that $b$ is an upper bound for $A$. 