Problem Set #4: Completeness Axiom (Part II)

Read: Chapter 5 on *Upper Bounds and Suprema,* pp. 80–85, in Michael J. Schramm's book: "Introduction to Real Analysis."

Problems:

1. Let $x \in \mathbb{R}$ and define $A_x = \{m \in \mathbb{Z} \mid m \leq x\}$.

   (a) Prove that A_x is non–empty.

   (b) Deduce from (a) that $\sup A_x$ exists and prove that there exist $n \in A_x$ such that
   $$\sup A_x < n + 1.$$

   (c) *(The Archimedean Property).* For any $x \in \mathbb{R}$ there exists $n \in \mathbb{Z}$ such that
   $$n \leq x < n + 1.$$

2. Use the Archimedean Property established in part (c) of Problem 1 in this Problem Set to prove the following statements.

   (a) For every $\varepsilon > 0$ there exists $n_o \in \mathbb{N}$ such that $0 < \dfrac{1}{n} < \varepsilon$ for all $n \in \mathbb{N}$ such that $n \geq n_o$.

   (b) For every x and y in $\mathbb{R}$ such that $x > 0$ and $y > 0$, there exists $n \in \mathbb{N}$ such that $y < nx$.

3. Let x and y be real numbers satisfying $x < y$.

   (a) Prove that there exists $m \in \mathbb{N}$ such that $m(y - x) > 1$.

   (b) With m as given by part (a), prove that there exists $n \in \mathbb{Z}$ such that $n \leqslant mx < n + 1$.

   (c) With m and n given by parts (a) and (b), show that $mx < n + 1 < my$, and deduce that there exists a rational number between x and y.

4. *(Density of $\mathbb{Q}$ in $\mathbb{R}$).* Prove that between any two real numbers there exits a rational number.

5. Prove that between any two real numbers there are infinitely many rational numbers.

6. Prove that between any two real numbers there exits an irrational number.

7. Let p be a positive real number. In this exercise we prove that there exists a real number x such that $x^2 = p$; that is, every positive real number has a square root.

   (a) Assume first that $p \geq 1$, and define $A = \{t \in \mathbb{R} \mid t > 0 \text{ and } t^2 \leq p\}$. Prove that $\sup A$ exists.

   (b) Let $s = \sup A$ and show that $s^2 = p$; that is, s is a solution of $x^2 = p$ for $p \geq 1$.

   (c) Let $0 < p < 1$. Prove that $x^2 = p$ has a solution in $\mathbb{R}$.

8. Prove that $\mathbb{Q}$ is not a complete ordered field.

9. For each of the following, (i) determine whether or not $\inf A$ and $\sup A$ exit, and (ii) compute them if they exist. In each case provide a justification for your answer.

   (a) $A = \mathbb{N}$.

   (b) $A = \left\{ \dfrac{n+1}{n} \;\Big|\; n \in \mathbb{N} \right\}$.

   (c) $A = \{x \in \mathbb{Q} \mid x^2 > 2\}$.

   (d) $A = \{x \in \mathbb{R} \mid 0 < x < 1\}$.

10. *(The Rational Root Theorem).* Let $a_o, a_1, a_2, \ldots, a_n$ be integers. If the equation

$$a_n x^n + a_{n-1} x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_o = 0$$

has a rational root $\dfrac{p}{q}$, expressed in lowest terms, then p divides a_o and q divides a_n.

11. Use the rational root theorem to prove that the following numbers are irrational.

   (a) $\sqrt{2} + \sqrt{3}$

   (b) $\sqrt[3]{2}$

   (c) $\dfrac{\sqrt{3}}{\sqrt[3]{2}}$