Problem Set #5: Completeness Axiom (Part III)

Read: Chapter 5 on *Upper Bounds and Suprema*, pp. 80–85, in Michael J. Schramm’s book: “Introduction to Real Analysis.”


Definitions and Notation.
A sequence of real numbers is a real–valued function defined on the set of natural numbers, \( f: \mathbb{N} \to \mathbb{R} \). For each \( n \in \mathbb{N} \), we write \( f(n) = x_n \), and denote the sequence \( f \) by its values \( (x_n) \). We say that a sequence \( (x_n) \) converges to \( x \in \mathbb{R} \) iff for every \( \varepsilon > 0 \) there exists \( n_o \in \mathbb{N} \), which depends on \( \varepsilon \), such that

\[
  n \geq n_o \implies |x_n - x| < \varepsilon.
\]

When \( \{x_n\} \) converges to \( x \) we write \( \lim_{n \to \infty} x_n = x \), and say that \( x \) is the limit of the sequence \( \{x_n\} \).

A sequence \( (x_n) \) is said to be increasing if \( x_n \leq x_{n+1} \) for all \( n \in \mathbb{N} \); the sequence \( (x_n) \) is said to be decreasing if \( x_n \geq x_{n+1} \) for all \( n \in \mathbb{N} \). A sequence \( (x_n) \) is said to be monotone if it is either increasing or decreasing. The sequence \( (x_n) \) is said to be bounded if there exists a real number \( M > 0 \) such that \( |x_n| \leq M \) for all \( n \in \mathbb{N} \).

Let \( (x_n) \) be a sequence of real numbers, and let \( (n_j) = (n_1, n_2, n_3, \ldots) \) be an increasing sequence of distinct natural numbers, then the sequence \( (x_{n_j}) \) is called a subsequence of \( (x_n) \).

A sequence \( (x_n) \) of real numbers is said to be a Cauchy sequence iff for every \( \varepsilon > 0 \) there exists \( n_o \in \mathbb{N} \), which depends on \( \varepsilon \), such that

\[
  n, m \geq n_o \implies |x_n - x_m| < \varepsilon.
\]

Problems:

1. Let \( (x_n) \) be a sequence of real numbers. Prove that if \( (x_n) \) converges, then its limit is unique.

2. [The Squeeze Theorem for Sequences] Let \( (x_n), (y_n) \) and \( (z_n) \) be sequences of real numbers. Suppose that there exists \( n_1 \in \mathbb{N} \) such that

\[
  x_n \leq y_n \leq z_n \quad \text{for all } n \geq n_1.
\]

Prove that if \( (x_n) \) and \( (z_n) \) converge and \( \lim_{n \to \infty} x_n = \lim_{n \to \infty} z_n = L \), then \( (y_n) \) converges to \( L \).
3. Prove that the following sequences converge and compute their limits.
   
   (a) \((x_n)\) where \(x_n = c\) for all \(n \in \mathbb{N}\) for a given \(c \in \mathbb{R}\).

   (b) \(\left(\frac{1}{n}\right)\)

   (c) \(\left(\frac{1}{2^n}\right)\)

4. Let \((x_n)\) and \((y_n)\) be sequences of real numbers. Suppose that \((x_n)\) and \((y_n)\) converge. Prove that the sequences \((x_n + y_n)\) and \((x_n \cdot y_n)\) also converge and compute their limits.

5. Let \((x_n)\) and \((y_n)\) be sequences of real numbers. Suppose that \(\lim_{n \to \infty} x_n = 0\) and that \((y_n)\) is bounded. Prove that \((x_n \cdot y_n)\) converges and compute its limit.

6. Prove that for any real number \(x\) there exists a sequence \((q_n)\) of rational values that converges to \(x\).

7. Let \((x_n)\) be a sequence of real numbers.
   
   (a) Prove that if \((x_n)\) is increasing and bounded from above, then \((x_n)\) converges.

   (b) Prove that if \((x_n)\) is decreasing and bounded from below, then \((x_n)\) converges.

   (c) Prove that any bounded and monotone sequence of real numbers must converge.

8. Let \((x_n)\) be a bounded sequence of real numbers. Prove that \((x_n)\) has a subsequence which converges.

9. Let \((x_n)\) be a sequence of real numbers. Prove that if \((x_n)\) converges, then it is a Cauchy sequence.

10. Prove that every Cauchy sequence of real numbers must be bounded.

11. Let \((x_n)\) be a Cauchy sequence of real numbers. Prove that if \((x_n)\) has a subsequence that converges to \(x \in \mathbb{R}\), then \(\{x_n\}\) converges and \(\lim_{n \to \infty} x_n = x\).

12. Prove that every Cauchy sequence of real numbers must converge.