Problem Set #5: Completeness Axiom (Part III)

Read: Chapter 5 on *Upper Bounds and Suprema,* pp. 80–85, in Michael J. Schramm's book: "Introduction to Real Analysis."

Read: Section 9.2 on *Convergence,* pp. 147–150, in Michael J. Schramm's book: "Introduction to Real Analysis."

Definitions and Notation.
A **sequence** of real numbers is a real–valued function defined on the set of natural numbers, $f\colon \mathbb{N} \to \mathbb{R}$. For each $n \in \mathbb{N}$, we write $f(n) = x_n$, and denote the sequence f by its values (x_n). We say that a sequence (x_n) **converges** to $x \in \mathbb{R}$ iff for every $\varepsilon > 0$ there exists $n_o \in \mathbb{N}$, which depends on ε, such that

$$n \geq n_o \implies |x_n - x| < \varepsilon.$$

When $\{x_n\}$ converges to x we write $\lim_{n\to\infty} x_n = x$, and say that x is the limit of the sequence $\{x_n\}$.

A sequence (x_n) is said to be **increasing** if $x_n \leqslant x_{n+1}$ for all $n \in \mathbb{N}$; the sequence (x_n) is said to be **decreasing** if $x_n \geqslant x_{n+1}$ for all $n \in \mathbb{N}$. A sequence (x_n) is said to be **monotone** if it is either increasing or decreasing. The sequence (x_n) is said to be **bounded** if there exists a real number $M > 0$ such that $|x_n| \leqslant M$ for all $n \in \mathbb{N}$.

Let (x_n) be a sequence of real numbers, and let $(n_j) = (n_1, n_2, n_3, \ldots)$ be an increasing sequence of distinct natural numbers, then the sequence (x_{n_j}) is called a **subsequence** of (x_n).

A sequence (x_n) of real numbers is said to be a **Cauchy sequence** iff for every $\varepsilon > 0$ there exists $n_o \in \mathbb{N}$, which depends on ε, such that

$$n, m \geqslant n_o \implies |x_n - x_m| < \varepsilon.$$

Problems:

1. Let (x_n) be a sequence of real numbers. Prove that if (x_n) converges, then its limit is unique.

2. *[The Squeeze Theorem for Sequences]* Let (x_n), (y_n) and (z_n) be sequences of real numbers. Suppose that there exists $n_1 \in \mathbb{N}$ such that

   $$x_n \leqslant y_n \leqslant z_n \quad \text{for all } n \geqslant n_1.$$

   Prove that if (x_n) and (z_n) converge and $\lim_{n\to\infty} x_n = \lim_{n\to\infty} z_n = L$, then (y_n) converges to L.

3. Prove that the following sequences converge and compute their limits.

   (a) (x_n) where $x_n = c$ for all $n \in \mathbb{N}$ for a given $c \in \mathbb{R}$.

   (b) $\left(\dfrac{1}{n}\right)$

   (c) $\left(\dfrac{1}{2^n}\right)$

4. Let (x_n) and (y_n) be sequences of real numbers. Suppose that (x_n) and (y_n) converge. Prove that the sequences $(x_n + y_n)$ and $(x_n \cdot y_n)$ also converge and compute their limits.

5. Let (x_n) and (y_n) be sequences of real numbers. Suppose that $\lim\limits_{n\to\infty} x_n = 0$ and that (y_n) is bounded. Prove that $(x_n \cdot y_n)$ converges and compute its limit.

6. Prove that for any real number x there exists a sequence (q_n) of rational values that converges to x.

7. Let (x_n) be a sequence of real numbers.

   (a) Prove that if (x_n) is increasing and bounded from above, then (x_n) converges.

   (b) Prove that if (x_n) is decreasing and bounded from below, then (x_n) converges.

   (c) Prove that any bounded and monotone sequence of real numbers must converge.

8. Let (x_n) be a bounded sequence of real numbers. Prove that (x_n) has a subsequence which converges.

9. Let (x_n) be a sequence of real numbers. Prove that if (x_n) converges, then it is a Cauchy sequence.

10. Prove that every Cauchy sequence of real numbers must be bounded.

11. Let (x_n) be a Cauchy sequence of real numbers. Prove that if (x_n) has a subsequence that converges to $x \in \mathbb{R}$, then $\{x_n\}$ converges and $\lim\limits_{n\to\infty} x_n = x$.

12. Prove that every Cauchy sequence of real numbers must converge.