Solutions to Assignment #1

1. Show that the solution to the difference equation

$$X_{t+1} = X_t$$

must be constant.

Solution: Let X_o denote the value of X_t at $t = 0$. It then follows that $X_1 = X_o$. Consequently, $X_2 = X_1 = X_o$. We claim that

$$X_n = X_o \quad \text{for all } n = 1, 2, 3, \ldots;$$

that is, X_n is constant. We prove the claim by induction on n. We have already seen that $X_1 = X_o$. Next, assume that $X_n = X_o$ and compute $X_{n+1} = X_n = X_o$. Hence, $X_{n+1} = X_o$. It then follows by the principle of mathematical induction that $X_t = X_o$ for all $t = 0, 1, 2, \ldots$ and therefore X_t is constant. □

2. *Modeling Red Blood Cell Production*[1]. In the circulatory system, red blood cells (RBCs) are constantly being filtered out and destroyed by specialized "clean–up" cells in the spleen and liver, and replenished by the bone marrow. Since the cells carry oxygen throughout the body, their numbers must be maintained at some constant level. In problems 2–5, we model the removing of RBCs by the spleen and liver, and their replenishing by the bone marrow in order to understand how the RBC levels may be maintained.

Assume that the spleen and liver remove a fraction f of the RBCs each day, and that the bone marrow produces new cells at a daily rate proportional to the number of RBCs lost on the previous day with proportionality constant γ.

Derive a system of two difference equations for R_t, the RBC count in circulation on day t, and M_t, the number of RBCs produced by the bone marrow on day t, where $t = 1, 2, 3, \ldots$

Solution: Consider the number of RBCs in circulation on day $t + 1$, R_{t+1}. By the *conservation principle,*

$$R_{t+1} - R_t = \text{Rate of RBCs in} - \text{Rate of RBCs out}$$

per unit of time, where

$$\text{Rate of RBCs in} = M_t$$

and

$$\text{Rate of RBCs out} = fR_t.$$

[1] Edelstein–Keshet, *Mathematical Models in Biology,* pg. 27

Thus,

$$R_{t+1} - R_t = M_t - fR_t,$$

from which we get that

$$R_{t+1} = (1 - f)R_t + M_t.$$

On the other hand, the number of new RBCs produced by the bone marrow on day $t + 1$, M_{t+1}, must be given by the expression

$$M_{t+1} = \gamma \cdot (\text{Number of RBCs removed on day } t) = \gamma(fR_t) = \gamma f R_t.$$

We then obtain the system of difference equations

$$\begin{cases} R_{t+1} &= (1-f)R_t + M_t \\ M_{t+1} &= \gamma f R_t \end{cases} \tag{1}$$

3. *Red Blood Cell Production (continued).* By considering the number of RBCs in circulation on day $t + 2$, we are able to combine the two difference equations derived in the previous problem into a single difference equation of the form

$$R_{t+2} = bR_{t+1} + cR_t, \tag{2}$$

where b and c are constants. Determine expressions for b and c in terms of f and γ.

Solution: By the first equation in (1),

$$R_{t+2} = (1 - f)R_{t+1} + M_{t+1}.$$

It then follows from the second equation in (1) that

$$R_{t+2} = (1 - f)R_{t+1} + \gamma f R_t.$$

This is in the form of (2) with $b = 1 - f$ and $c = \gamma f$. □

4. *Red Blood Cell Production (continued).* We may seek to find a solution to the linear second order difference equation (2) as follows:

   (a) Assume that the sought after solution is of the form $R_t = A\lambda^t$, where A is some constant that will depend on the initial conditions, and λ is a parameter that is to be determined by substituting into the difference equation. Substitute this assumed form for R_t into equation (2) to obtain an expression for λ. Assuming that neither A nor λ are zero, simplify the expression to get the second order equation

$$\lambda^2 = b\lambda + c. \tag{3}$$

Solution: Substitute $R_t = A\lambda^t$ into (2) to get that

$$A\lambda^{t+2} = bA\lambda^{t+1} + cA\lambda^t$$

or

$$A\lambda^{t+2} - bA\lambda^{t+1} - cA\lambda^t = 0$$

Factoring out the common term $A\lambda^t$ on the left–hand side of the previous equation leads to

$$A\lambda^t(\lambda^2 - b\lambda - c) = 0.$$

Since A and λ are not both zero, it follows that

$$\lambda^2 - b\lambda - c = 0, \tag{4}$$

which leads to (3). □

(b) Solve equation (3) for λ to obtain two possible solutions λ_1 and λ_2 in terms of f and γ, where $\lambda_1 < \lambda_2$.

Solution: To solve (3), apply the quadratic formula to (4) to obtain

$$\lambda = \frac{b \pm \sqrt{b^2 + 4c}}{2}.$$

Thus,

$$\lambda_1 = \frac{b - \sqrt{b^2 + 4c}}{2} = \frac{1 - f - \sqrt{(1-f)^2 + 4\gamma f}}{2}$$

and

$$\lambda_2 = \frac{b + \sqrt{b^2 + 4c}}{2} = \frac{1 - f + \sqrt{(1-f)^2 + 4\gamma f}}{2}. \quad \square$$

(c) Verify that $A_1\lambda_1^t$ and $A_2\lambda_2^t$, where A_1 and A_2 are arbitrary constants, both solve the difference equation (2).

Solution: Let $R_t = A_1\lambda_1^t$ and compute

$$\begin{aligned} R_{t+2} - bR_{t+1} - cR_t &= A_1\lambda_1^{t+2} - bA_1\lambda_1^{t+1} - cA_1\lambda_1^t \\ &= A_1\lambda_1^t(\lambda_1^2 - b\lambda_1 - c) \\ &= A_1\lambda_1^t(0) = 0 \end{aligned}$$

since λ_1 solves the quadratic equation (4). Thus, $R_t = A_1\lambda_1^t$ solves (2). The same argument also shows that $R_t = A_2\lambda_2^t$ also solves (2). □

(d) Verify that

$$R_t = A_1\lambda_1^t + A_2\lambda_2^t, \tag{5}$$

where A_1 and A_2 are arbitrary constants, also solves the difference equation (2).

Solution: Consider

$$\begin{aligned} R_{t+2} - bR_{t+1} - cR_t &= A_1\lambda_1^{t+2} + A_2\lambda_2^{t+2} - b(A_1\lambda_1^{t+1} + A_2\lambda_2^{t+2}) - c(A_1\lambda_1^t + A_2\lambda_2^t) \\ &= A_1\lambda_1^{t+2} - bA_1\lambda_1^{t+1} - cA_1\lambda_1^t + A_2\lambda_2^{t+2} - bA_2\lambda_2^{t+1} - cA_2\lambda_2^t \\ &= A_1\lambda_1^t(\lambda_1^2 - b\lambda_1 - c) + A_2\lambda_2^t(\lambda_2^2 - b\lambda_1 - c) \\ &= A_1\lambda_1^t(0) + A_2\lambda_2^t(0) \end{aligned}$$

since both λ_1 and λ_2 solve the quadratic equation (4). Thus,

$$R_{t+2} - bR_{t+1} - cR_t = 0$$

which shows that $R_t = A_1\lambda_1^t + A_2\lambda_2^t$ also solves (2). □

5. *Red Blood Cell Production (continued).* Assume that 1% of the RBCs are filtered out of circulation by the spleen and liver in a day; that is $f = 0.01$.

(a) If $\gamma = 1.50$, what does the general solution (5) predict about the RBC count as $t \to \infty$?

Solution: In this case $\lambda_1 \approx -0.03$ and $\lambda_2 \approx 1.005$, so that, since $|\lambda_1| < 1$ and $\lambda_2 > 1$, it follows that

$$\lim_{t\to\infty} R_t = +\infty.$$

That is, the RBC count increases without bound.

(b) Suppose now that $\gamma = 0.50$. What does the general solution (5) predict about the RBC count as $t \to \infty$?

Solution: Here, $\lambda_1 \approx -0.005$ and $\lambda_2 \approx 0.995$, so that, since both $|\lambda_1|$ and λ_2 are strictly less than 1, it follows that

$$\lim_{t\to\infty} R_t = 0.$$

That is, the RBC count will eventually go to zero.

(c) Suppose now that $\gamma = 1$. What does the general solution (5) predict about the RBC count as $t \to \infty$?

Solution: In this last case, $\lambda_1 = -f = -.01$ and $\lambda_2 = 1$; thus, since $|\lambda_1| < 1$,

$$\lim_{t\to\infty} R_t = A_2.$$

That is, the RBC count will tend towards the constant value A_2 ar $t \to \infty$.

(d) Which of the three values of γ discussed in the previous three parts seems to yield a reasonable prediction? What implication does that have about RBC levels in the long run?

Solution: Only the third case, namely $\gamma = 1$, will yield a situation in which the RBC levels may be maintained.