## Solutions to Assignment #2

1. Consider the population model given by the difference equation

$$N_{t+1} - N_t = m,$$

where $m$ is a constant, for $t = 0, 1, 2, \ldots$.

   (a) Give an interpretation for this model.

   *Solution:* This equation says that the population increment (or decrease, if $m < 0$) is constant; equivalently, after each unit of time, the same number of individuals are added (or taken away from) to the population. □

   (b) If the initial population density is $N_o$, what does this model predict in the long run? Consider the two possibilities $m < 0$ and $m > 0$.

   *Solution:* From $N_{t+1} = N_t + m$ we get tht $N_1 = N_o + m$. Consequently, $N_2 = N_1 + m = N_o + m + m = N_o + 2m$. Similarly, $N_3 = N_o + 3m$. Thus, it follows by induction on $n$ that $N_n = N_o + nm$ for all $n = 1, 2, 3, \ldots$ Hence

$$N_t = N_o + mt \quad \text{for all } \ t = 1, 2, 3 \ldots$$

   Hence, if $m > 0$, the population will increase linearly and indefinitely, while if $m < 0$, it will decrease to extinction in a finite time. □

   (c) How does this model compare with the Malthusian model?

   *Solution:* This model predits linear growth or decay, while the Malthusian model predits geometric growth or decay. □

2. Assume that the *per–capita* growth rate $\lambda$ of a population is less than 1; that is, left on its own, the population will go extinct. To avoid extinction, suppose that after each unit of time, a constant number $m$ of individuals of the same species is added to the population.

   (a) Write down a difference equation that models this situation.

   *Solution:* $N_{t+1} = \lambda N_t + m$.

   (b) Solve the difference equation and discuss what this model predicts in the long run.

   *Solution:* Suppose that at time $t = 0$ there are $N_o$ individuals. Then, $N_1 = \lambda N_o + m$. Thus, $N_2 = \lambda N_1 + m = \lambda(\lambda N_o + m) + m = \lambda^2 N_o + \lambda m + m$.

In a similar manner we can compute $N_3 = \lambda^3 N_o + \lambda^2 m + \lambda m + m$. Hence, by induction on $n$ we can show that

$$\begin{aligned} N_n &= \lambda^n N_o + \lambda^{n-1} m + \lambda^{n-2} m + \cdots + \lambda m + m \\ &= N_o \lambda^n + m\left(\lambda^{n-1} + \lambda^{n-2} + \cdots + \lambda + 1\right) \\ &= N_o \lambda^n + m \cdot \frac{\lambda^n - 1}{\lambda - 1} \end{aligned}$$

for $n = 1, 2, 3, \ldots$. Consequently,

$$N_t = N_o \lambda^t + m \cdot \frac{1 - \lambda^t}{1 - \lambda} \quad \text{for} \quad t = 0, 1, 2, \ldots$$

Now, since $|\lambda| < 1$ it follows that

$$\lim_{t \to \infty} N_t = \frac{m}{1 - \lambda}.$$

Thus, this model predicts that the population will tend to the equilibrium value of $m/(1-\lambda)$ □

(c) How does this model compare with the Malthusian model?

*Solution:* While the Malthusian model (with $\lambda < 1$) predicts extinction, this model predicts that the population will tend towards a non-zero steady state. □

3. *[Problem 1.1.2 on page 6 in Allman and Rhodes].* In early stages of the development of a frog embryo, cell division at a fairly regular rate. Suppose that you observe that all cells divide, and hence the number of cells doubles, roughly every half hour.

(a) Write down an equation modeling this situation.

*Solution:* Let $N_t$ denote the number of cells in the embryo at time $t$, where $t$ denotes the number of doubling times; that is, $t$ is measured in mumbers of 30–minute periods. Assume also that there is one cell ($N_o = 1$) at the start of the process. Then, the difference equation modeling the growth of the embryo is

$$N_{t+1} = 2N_t. \quad \square$$

(b) Produce a table and graph the number of cells in the embryo as a function of $t$.

*Solution:* Figure 2 shows the graph. □

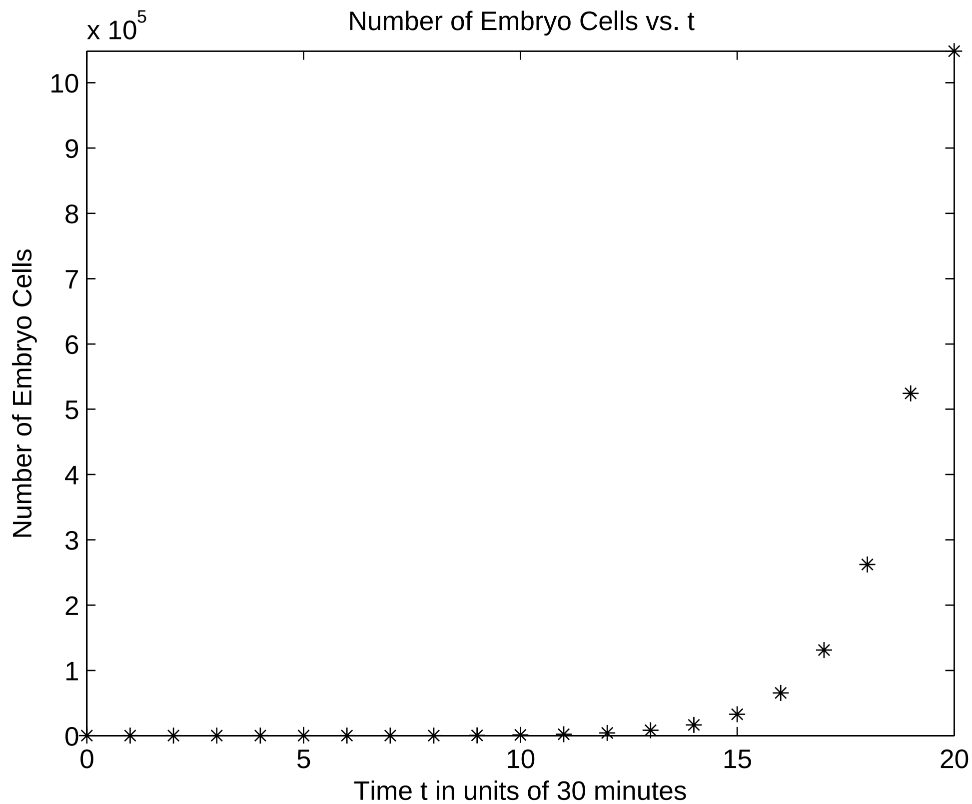


Figure 1: Graph for Problem 1.1.2 part (b)

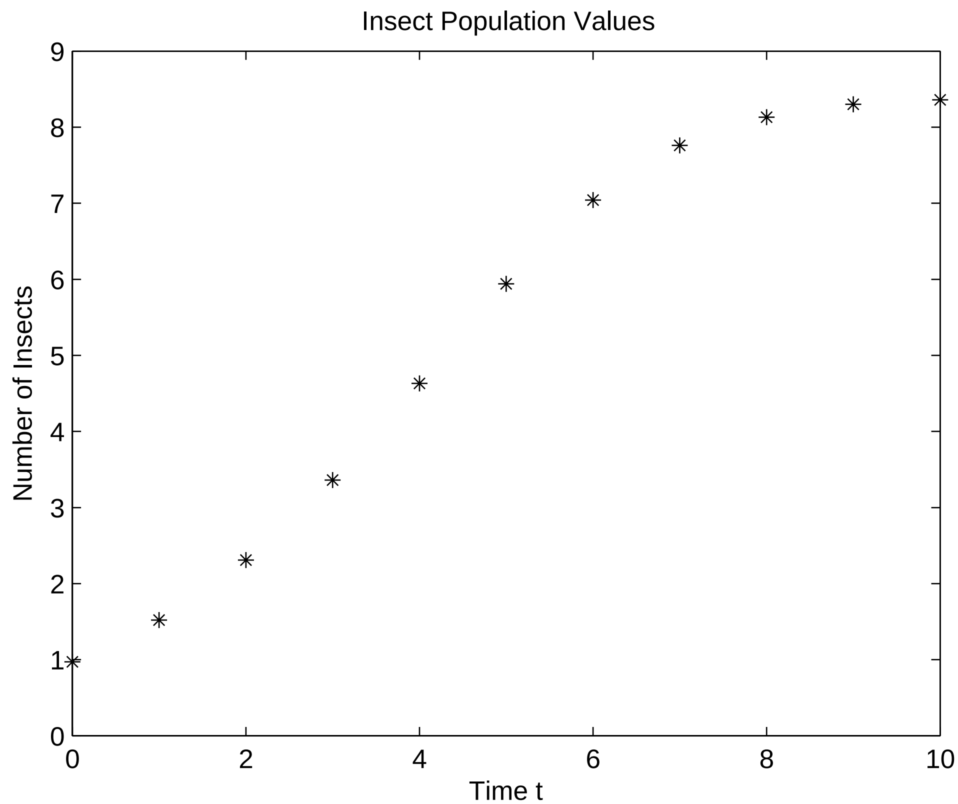


Figure 2: Plot of Insect Population Values on p. 7 of Allman and Rhodes

(c) Further observation shows that, after 10 hours, the embryo has 30,000 cells. Is this roughly consistent with the model? What biological conclusions and/or questions does this raise?

*Solution:* A time period of 10 hours corresponds to $t = 20$. The predicted value then is $N_10 = 2^{20} = 1,048,576$. There is therefore a large discrepancy suggesting that a simple geometric growth model is not the appropriate one for embryo cells. Perhaps, after several divisions, cells especialize and differentiate and therefore might take longer to divide. □

4. *[Problem 1.1.6 on page 7 in Allman and Rhodes].*

*Solution:* Figure 3 shows the graph of the insects population values versus $t$ in Table 1.2 on p. 7 of Allman and Rhodes. Insect growth is definitely not consistent with the geometric growth model. Perhaps, this might be the case over the time interval $[0, 4]$. However, the logistic model seems to be more appropriate in this case. □

5. *[Problem 1.1.10 on page 7 in Allman and Rhodes].* A model for the growth of $P_t$ is said to have a *steady state* or *equilibrium point* at $P^*$ if whenever $P_t = P^*$, then $P_{t+1} = P^*$.

   (a) This is equivalent to saying that: $P^*$ is a steady state if, whenever $P_t = P^*$, then $\Delta P = 0$. □

   (b) More intuitively, $P^*$ is a steady state, if whenever the value $P^*$ is reached, the population values remain at $P^*$ for all values of $t$. □

   (c) Can a model described by $P_{t+1} = (1 + r)P_t$ have a steady state? Explain.

   *Solution:* Suppose there is a steady state $P^*$. It then follows that $P^* = (1 + r)P^*$, which implies that $1 = 1 + r$, and therefore $r = 0$. Thus, there is a steady state only when $r = 0$. Notice that in this case we get the difference equation $P_{t+1} = P_t$ which can only have constant solutions. □