Solutions to Assignment #5

1. Suppose that X_t satisfies the *difference inequlity*

$$|X_{t+1}| \leq \eta |X_t| \quad \text{for} \quad t = 0, 1, 2, 3, \ldots$$

where $0 < \eta < 1$. Prove that $\lim_{t\to\infty} X_t = 0$.

Solution: For $t = 0$ we get

$$|X_1| \leq \eta |X_o|.$$

Similarly, for $t = 1$, we get

$$|X_2| \leq \eta |X_1| \leq \eta^2 |X_o|,$$

by the previous inequality. We may, therefore, proceed by induction on n to prove that

$$|X_n| \leq \eta^n |X_o| \quad \text{for} \quad n = 1, 2, 3, \ldots$$

We therefore have that

$$0 \leqslant |X_t| \leqslant \eta^t |X_o|, \qquad \text{for } t = 0, 1, 2, \ldots,$$

where $0 < \eta < 1$, so that

$$\lim_{t\to\infty} \eta^t = 0.$$

It then follows by the Squeeze Theorem, or the Sandwich Theorem, that

$$\lim_{n\to\infty} |X_n| = 0.$$

Hence, $\lim_{t\to\infty} X_t = 0$. □

2. The *Principle of Linearized Stability* for the difference equation

$$N_{t+1} = f(N_t)$$

states that, if f is differentiable at a fixed point N^* and

$$|f'(N^*)| < 1,$$

then N^* is an assymptotically stable equilibrium solution.

In this problem we use the Principle of Linearized stability to analyze the following population model:

$$N_{t+1} = \frac{kN_t}{b + N_t}$$

where k and b are postive parameters.

(a) Write the model in the form $N_{t+1} = f(N_t)$ and give the fixed points of f. What conditions of k and b must be imposed in order to ensure that the model will have a non–negative steady state?

Solution: $f(x) = \dfrac{kx}{b+x}$ in this case, so that the fixed points of f are solutions to the equation

$$\frac{kx}{b+x} = x,$$

or

$$\frac{kx}{b+x} - x = 0.$$

Factoring the last expression we get

$$x\left(\frac{k}{b+x} - 1\right) = 0.$$

Thus, either $x = 0$ or $\dfrac{k}{b+x} - 1 = 0$. Solving the last expression for x we obtain $x = k - b$. Thus, the fixed point of f are

$$N^* = 0 \quad \text{and} \quad N^* = k - b.$$

For the second fixed point to be nonnegative, it must be the case that $b \leq k$. $\square$

(b) Determine the stability of the equilibrium points found in part (a).

Solution: We apply the Principle of Linearized Stability. Compute

$$f'(x) = \frac{bk}{(b+x)^2}.$$

Then, $f'(0) = \dfrac{bk}{b^2} = \dfrac{k}{b} \geq 1$ since $b \leq k$, by part (a). Thus, if $b < k$, then $N^* = 0$ is unstable, by the Principle of Linearized Stability. If if $b = k$, the Principle of Linearized Stability does not apply.

Similarly, since $f'(k-b) = \dfrac{bk}{k^2} = \dfrac{b}{k} \leq 1$ since $b \leq k$, by part (a). Thus, if $b < k$, then $N^* = k - b$ is asymptotically stable, by the Principle of Linearized Stability. On the other hand, if $b = k$, the Principle of Linearized Stability does not apply. $\square$

3. *[Problems 1.3.6 (d)(e) on page 29 in Allman and Rhodes]*

(d) Determine the equilibrium points of $\Delta P = aP - bP^2$.

Solution: Solve the equation $aP - bP^2 = 0$, $P(a - bP) = 0$ to obtain $P^* = 0$ or $P^* = a/b$ (here we are assuming that $b \neq 0$). □

(e) Determine the equilibrium points of $P_{t+1} = cP_t - dP_t^2$.

Solution: Here we find the fixed points of $f(P) = cP - dP^2$; that is, we solve the equation $f(P) = P$, or $cP - dP^2 = P$. To solve this equation, we rewrite it as

$$(c-1)P - dP^2 = 0,$$

from which we get, after factoring that

$$P[(c-1) - dP] = 0.$$

Thus, $P^* = 0$ or $P^* = (c-1)/d$, for $d \neq 0$. □

4. *[Problems 1.3.7 (d)(e) on page 29 in Allman and Rhodes]* For each of the equations in the previous problem, use the principle of linearized stability to determine the stability of each of the equilibrium points.

(d) $\Delta P = aP - bP^2$.

Solution: Here, $f(P) = P + aP - bP^2$, so that $f'(P) = 1 + a - 2bP$. Thus, $f'(0) = 1 + a$. Hence, $P^* = 0$ is stable for $-2 < a < 0$, and unstable for $a > 0$ or $a < -2$.

Similarly, since $f'(a/b) = 1 + a - 2b(a/b) = 1 - a$, $P^* = a/b$ is stable for $0 < a < 2$, and unstable for $a < 0$ or $a > 2$. □

(e) $P_{t+1} = cP_t - dP_t^2$.

Solution: In this case, $f(P) = cP - dP^2$ and so $f'(P) = c - 2dP$.

Thus, $f'(0) = c$ and so $P* = 0$ is stable if $|c| < 1$ and unstable if $|c| > 1$.

Similarly, since $f'((c-1)/d) = 2 - c$, $P^* = (c-1)/d$ is stable is $1 < c < 3$, and unstable if $c < 1$ or $c > 3$. □

5. Problems 1.3.11 (a)(b)(c)(d) on page 30 in Allman and Rhodes.

Note: The code for the MATLAB® program `onepop` *may be downloaded from the courses website at* `http://pages.pomona.edu/~ajr04747`.

Many biological processes involve *diffusion.* A simple example is the passage of oxygen from the from the lung into the bloodstream (and the passage of carbon dioxide in the opposite direction). A simple model views the lung as a single compartment with oxygen concentration L and the bloodstream an adjoining compartment with oxygen concentration B. If, for simplicity, we assume that the compartments both have volume 1, then in the time span of a single breath the total oxygen $K = L + B$ is constant. If we think of a very *small* time interval, then the increase of B over this time interval will be proportiaonal to the difference between L and B. That is,

$$\Delta B = r(L - B). \tag{1}$$

(This experimental fact is sometimes called *Fick's Law.*)

(a) In what range must the parameter r be for this model to be meaningful?

Solution: $0 < r < 1$ since (i) the oxigen concentration in the bloodstream must increase (with oxygen coming from the lungs) if $L > B$, and decrease of $B > L$; and (ii) even if B is very low, it can not increase by an amount larger than the amount of oxygen available in the lungs. □

(b) Use the fact that $L + B = K$ to write the model (1)using only the parameters r and K to describe ΔB in terms of B.

Solution: Solving for L in $L + B = K$ and susbtituting into (1 yields

$$\Delta B = r(K - 2B). \qquad \square$$

(c) For $r = 0.1$ and $K = 1$, and a variety of choices for B_o, investigate the MATLAB® program `onepop`. How do things change is a different valueof r is used?

Solution: For any initial condition B_o, the solutions tend to $K/2 = 0.5$ as $t \to \infty$. The result is the same for any r with $0 < r < 1$. □

(d) Algebraically, find the equilibrium point B^* for (1. Does this fit with what you saw in part (c)? Can you explain this result intuitively?

Solution: We apply the Principle of Linearized Stability. In this case $f(B) = B + r(K - 2B)$, so that the the fixed point of f is B such that $f(B) = B$, which yields $B^* = K/2$. To determine whether or not B^* is stable, compute $f'(B) = 1 - 2r$. Thus, $B^* = K/2$ is stable if $|1 - 2r| < 1$ or $0 < r < 1$. This is precisely what we saw in the numerical experiments in part (c). Intuitevly, as time goes on, after many breaths, the oxygen concentration in the bloodstream should reach a steady state which is equal the amoung of oxygen in the lungs. □