Solutions to Assignment #6

1. Suppose the growth of a population is governed by the *differential equation*

$$\frac{dN}{dt} = -kN$$

where k is a positive constant.

(a) Explain why this model predicts that the population will decrease as time increases.

Solution: Since $k > 0$, $N'(t) = -kN < 0$ for all t and all $N > 0$. It then follows that $N(t)$ will always decrease. □

(b) If the population at $t = 0$ is N_o, find the time t, in terms of k, at which the population will be reduced by half.

Solution: Solving the equation subject to the initial condition $N(0) = N_o$ yields $N(t) = N_o e^{-kt}$. We want t such that $N(t) = N_o/2$; that is

$$N_o e^{-kt} = \frac{N_o}{2} \quad \text{or} \quad e^{-kt} = \frac{1}{2}.$$

Solving for k yields

$$t = -\frac{1}{k} \ln\left(\frac{1}{2}\right) = \frac{\ln(2)}{k}. \quad \square$$

2. Consider a bacterial population whose relative growth rate is given by

$$\frac{1}{N}\frac{dN}{dt} = K$$

where $K = K(t)$ is a continuous function of time, t.

(a) Suppose that $N_o = N(0)$ is the initial population density. Verify that

$$N(t) = N_o \exp\left(\int_0^t K(\tau)\, \mathrm{d}\tau\right)$$

solves the differential equation and satisfies the initial condition.

Solution: First observe that

$$N(0) = N_o \exp\left(\int_0^0 K(\tau)\, \mathrm{d}\tau\right) = N_o \exp(0) = N_o,$$

and so $N(t)$ satisfies the initial condition. To show that $N(t)$ solves the differential equation, use the Chain Rule and the Fundamental Theorem of Calculus to get

$$\begin{aligned} N'(t) &= N_o \exp\left(\int_0^t K(\tau)\, \mathrm{d}\tau\right) \cdot \frac{d}{dt}\left(\int_0^t K(\tau)\, \mathrm{d}\tau\right) \\ &= N_o \exp\left(\int_0^t K(\tau)\, \mathrm{d}\tau\right) \cdot K(t) \\ &= K(t)N(t); \end{aligned}$$

that is, $\dfrac{dN}{dt} = KN.$ □

(b) Find $N(t)$ if $K(t) = \begin{cases} 1-t & \text{if } 0 \leq t \leq 1 \\ 0 & \text{if } t > 1 \end{cases}$. Sketch the graph of $N(t)$.

Solution: Compute $\displaystyle\int_0^t K(\tau)d\tau = \begin{cases} t - t^2/2 & \text{if } 0 \leq t \leq 1 \\ 1/2 & \text{if } t > 1 \end{cases}$. Then,

$$N(t) = \begin{cases} N_o e^{t-t^2/2} & \text{if } 0 \leq t \leq 1 \\ N_o e^{1/2} & \text{if } t > 1 \end{cases}.$$

Figure 1 shows a sketch of this solution for the case $N_o = 1$.

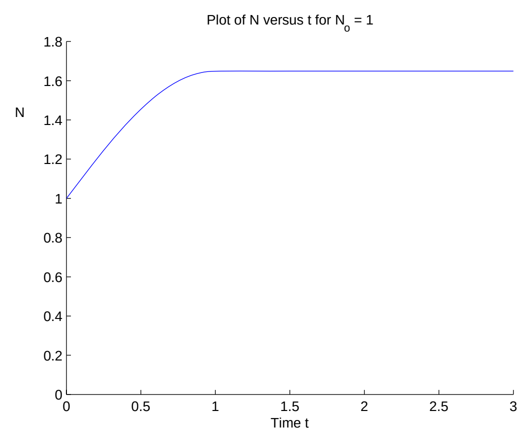

Figure 1: Graph of $N(t)$ in Problem (2)(b)

3. For any population (ignoring migration, harvesting, or predation) one can model the relative growth rate by the following conservation principle

$$\frac{1}{N}\frac{dN}{dt} = \text{birth rate (per capita)} - \text{death rate (per capita)} = b - d,$$

where b and d could be functions of time and the population density N.

(a) Suppose that b and d are linear functions of N given by $b = b_o - \alpha N$ and $d = d_o + \beta N$ where b_o, d_o, α and β are positive constants. Assume that $b_o > d_o$. Sketch the graphs of b and d as functions of N. Give a possible interpretation for these graphs.

Solution: This model assumes that the *per capita* birth and death rates are linear functions o the population density, N. The birth rate decreases with increasing N, while the death rate increases with increasing N. The population size, K, for which both rates are the same gives an equilibrium point. □

(b) Find the point where the two lines sketched in part (a) intersect. Let K denote the first coordinate of the point of intersection. Show that

$$K = \frac{b_o - d_o}{\alpha + \beta}.$$

K is the carrying capacity of the population.

(c) Show that $\dfrac{dN}{dt} = rN\left(1 - \dfrac{N}{K}\right)$ where $r = b_o - d_o$ is the intrinsic growth rate.

4. The following equation models the evolution of a population that is being harvested at a constant rate:

$$\frac{dN}{dt} = 2N - 0.01N^2 - 75.$$

Find equilibrium solutions and sketch a few possible solution curves. According to model, what will happen if at time $t = 0$ the initial population densities are 40, 60, 150, or 170?

Solution: The equilibrium points are 50 and 150. The first one is unstable, while the second one is stable. If the initial population density is below 50 the population will go extinct in finite time. If the initial population is above 50, solutions will tend towards the stable equilibrium point at 150.

Figure 2 shows a sketch generated by the MATLAB® program `dfield.m`. □

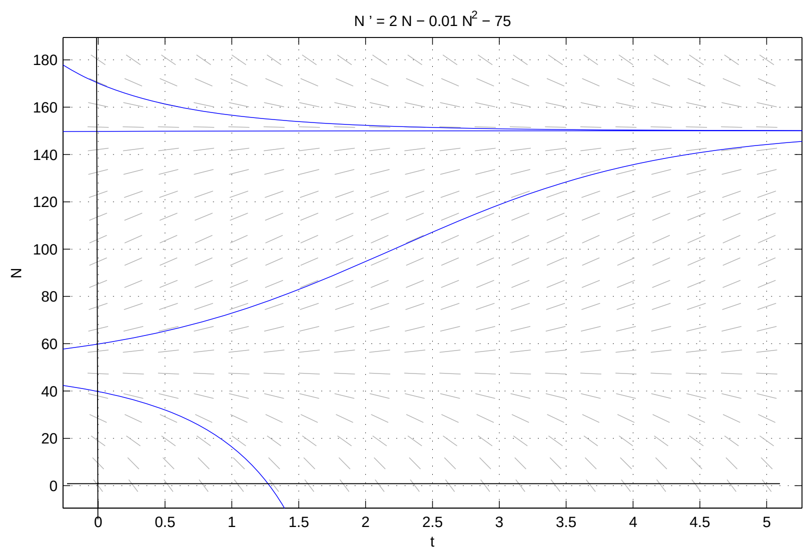

Figure 2: Solution Curves for Problem (4)

5. Consider the modified logistic model

$$\frac{dN}{dt} = rN\left(1 - \frac{N}{K}\right)\left(\frac{N}{T} - 1\right)$$

where $N(t)$ denotes the population density at time t, and $0 < T < K$.

(a) Find the equilibrium solutions and determine the nature of their stability.
Solution: Equilibrium solutions are $\overline{N}_1 = 0$, $\overline{N}_2 = T$ and $\overline{N}_2 = K$. $\overline{N}_2$ is unstable, while $\overline{N}_1$ and $\overline{N}_3$ are asymptotically stable. □

(b) Sketch other possible solutions to the equation.
Solution: Figure 3 shows a sketch generated by the MATLAB®program `dfield.m` for the case $r = 1$, $K = 2$ and $T = 1$. □

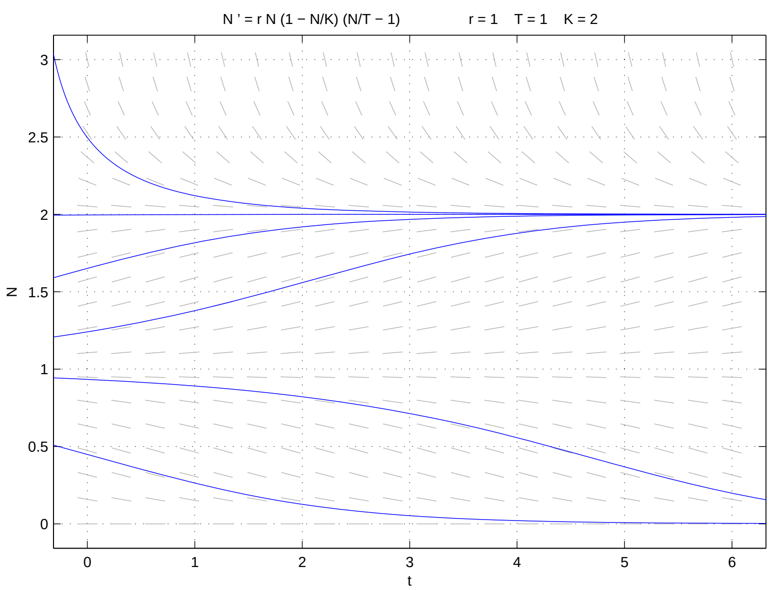

Figure 3: Solution Curves for Problem (5)(b)

(c) Describe what the model predicts about the population and give a possible explanation.
Solution: The model predicts that if the initial population size is below the threshold value T, the population will eventually go extinct. If the initial population value is above the threshold value, the population will tend towards the carrying capacity value of K. □