Solutions to Assignment #7

1. Suppose a bacterial colony is growing according to the (continuous) Malthusian model. If the time, t, is measured in units of *division cycle* divided by $\ln 2$, give a formula for $N(t)$, given that $N(0) = N_o$. By how much does the population increase in one unit of time? (*Note:* A *division cycle* would correspond to the doubling time.)

> ***Solution:*** Assume first the $N = N(\tau)$, where τ is measured in an arbitrary continuous time unit. Then, the solution to the Malthus differential equation
> $$\frac{dN}{d\tau} = kN,$$
> subject to $N(0) = N_o$, is given by
> $$N(t) = N_o e^{k\tau}.$$
> If T is the doubling time, then $k = \dfrac{\ln(2)}{T}$ and so
> $$N(\tau) = N_o e^{\frac{\ln(2)}{T}\tau} = N_o e^{\frac{\tau}{T/\ln(2)}}.$$
> Thus, if t counts the number of division cycles divided by $\ln(2)$, it follows that $t = \dfrac{\tau}{T/\ln(2)}$; and therefore
> $$N(t) = N_o e^t.$$
> Thus, in one division cycle divided by $\ln(2)$, the population increases by
> $$\frac{N(1) - N(0)}{N_o} = \frac{N_o e - N_o}{N_o} = e - 1 \approx 1.718$$
> or about 172%. □

A Conservation Principle for a One–Compartment Model. Suppose you are tracking the amount, $Q(t)$, of a substance in some predefined space or region, known as a *compartment,* at time t. (A compartment could represent, for instance, the bloodstream in the body of a patient, and $Q(t)$ the amount of a drug present in the bloodstream at time t). If we know, or can model, the rates at which the substance enters or leaves the compartment, then the rate of the

change of the substance in the compartment is determined by the differential equation:

$$\frac{dQ}{dt} = \text{Rate of substance in} - \text{Rate of substance out}, \hspace{2cm} (1)$$

where we are assuming that Q is a differentiable function of time.

2. Assume that the rate at which a drug leaves the bloodstream and passes into the urine is proportional to the quantity of the drug in the blood at that time.

   (a) Use the conservation principle (1) to write down a differential equation for the quantity, Q, of the drug in the blood at time, t, in hours.

   > **Solution**: Assume that, after an initial dose of Q_o, not additional drug is injected into the patient. Then, Rate of Q in $= 0$ and Rate of Q out $= kQ$, where $k > 0$ is a constant of proportionality. Thus, the conservation principle (1) implies that
   >
   > $$\frac{dQ}{dt} = -kQ.$$
   >
   > $\hspace{12cm} \square$

   (b) Solve the differential equation derived in part (a) for the case in which an initial dose of Q_o is injected directly into the blood.

   > **Answer**: $Q(t) = Q_o e^{-kt}$ for all t. $\hspace{5cm} \square$

   (c) Assume that 20% of the initial dose is left in the blood after 3 hours. Write a formula for computing $Q(t)$ for any time t, in hours.

   > **Solution**: If $Q(3) = 0.20 \; Q_o$, then
   >
   > $$Q_o e^{-3k} = 0.20 \; Q_o \quad \text{or} \quad e^{-3k} = \frac{1}{5}.$$
   >
   > Thus, $-k = \dfrac{1}{3} \ln\left(\dfrac{1}{5}\right)$, and so the solution is given by
   >
   > $$Q(t) = Q_o e^{\ln\left(\frac{1}{5}\right)\frac{t}{3}} = Q_o \left(\frac{1}{5}\right)^{t/3}$$
   >
   > for all t. $\hspace{11cm} \square$

   (d) What percentage of the initial dosage of the drug is left in the patient's body after 6 hours?

Solution: After 6 hours, the amount of drug present in the patient's blood is

$$Q(6) = Q_o \left(\frac{1}{5}\right)^{6/3} = \frac{1}{25}Q_o,$$

or 4% of the initial dose. ☐

3. A patient is given the drug theophylline intravenously at a constant rate of 43.2 mg/hour to relieve acute asthma. You can imagine the drug as entering a compartment of volume 35,000 ml. (This is an estimate of the volume of the part of the body through which the drug circulates.) The rate at which the drug leaves the patient is proportional to the quantity there, with proportionality constant 0.082.

   (a) Write a differential equation for the quantity, $Q = Q(t)$, in milligrams, of the drug in the body at time t hours.

   Solution: Use the conservation principle stated in equation (1) for a one–compartment model to get

   $$\frac{dQ}{dt} = \text{Rate of } Q \text{ in} - \text{Rate of } Q \text{ out}$$
   $$= 43.2 - 0.082Q.$$

   Thus,

   $$\frac{dQ}{dt} = -k(Q - \overline{Q}),$$

   where $k = 0.082$ and $\overline{Q} = \dfrac{43.2}{0.082} \doteq 526.83.$ ☐

   (b) Give the equilibrium solution, $\overline{Q}$, to the equation in part (a).

   Answer: $\overline{Q} = \dfrac{43.2}{0.082} \doteq 526.83.$ ☐

   (c) Assuming that the patient's body contains none of the drug initially, give $Q(t)$ for all t, and sketch an approximate graph of Q as a function of t.
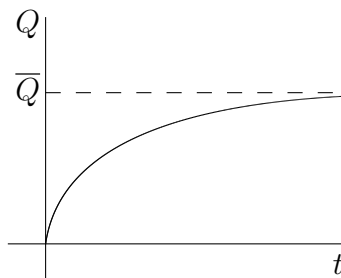
   Solution: Use separation of variables to get

   $$Q(t) = \overline{Q} + ce^{-kt},$$

   for some constant c. Since $Q(0) = 0$, we get that

   $$Q(t) = \overline{Q}(1 - e^{-kt}).$$

   A sketch of the graph of this function is shown in Figure 1. ☐

Figure 1: Sketch of graph of $Q(t)$

(d) What is the limiting value of $Q(t)$ as $t \to \infty$?

      Solution:
$$\lim_{t \to \infty} Q(t) = \lim_{t \to \infty} \overline{Q}(1 - e^{-kt}) = \overline{Q}.$$

                □

4. When people smoke, carbon monoxide is released into the air. Suppose that in a room of volume 60 m³, air containing 5% carbon monoxide is introduced at a rate of 0.002 m³/min. (This means that 5% of the volume of incoming air is carbon monoxide). Assume that the carbon monoxide mixes immediately with the air and the mixture leaves the room at the same rate as it enters.

(a) Let $Q = Q(t)$ denote the volume (in cubic meters) of carbon monoxide in the room at any time t in minutes. Use the conservation principle (1) to write down a differential equation for Q.

      Solution: Imagine the room as a compartment of, fixed volume, V. In this case, $V = 60$ m³. Air flows into the room at rate, F, of 0.002 cubic meters per minute. The air that flows into the room has a concentration, c_i, of carbon monoxide, where $c_i = 5\%$ (the concentration here is measured in percent volume). Let $Q(t)$ denote the amount of carbon monoxide present in the room at time t. The conservation principle in (1) in this case takes the form

$$\frac{dQ}{dt} = \text{Rate of } Q \text{ in} - \text{Rate of } Q \text{ out},$$

      where

$$\text{Rate of } Q \text{ in} = c_i F,$$

and

$$\text{Rate of } Q \text{ out} = c(t)F,$$

where $c(t) = \dfrac{Q(t)}{V}$ is the concentration of carbon monoxide in the room at time t. Here we are assuming that the volume, V, of air in the room is fixed, so that the rate of flow of air into the room is the same as the rate of flow out of the room.

We then have that

$$\frac{dQ}{dt} = c_i F - \frac{Q(t)}{V}F,$$

which we can write as

$$\frac{dQ}{dt} = -\frac{F}{V}(Q(t) - c_i V). \tag{2}$$

Putting in the values of F, V and c_i we obtain

$$\frac{dQ}{dt} = -\frac{1}{3} \times 10^{-3}(Q(t) - 3)$$

in units of cubic meters per minute.   □

(b) Give the equilibrium solution, $\overline{Q}$, to the differential equation in part (a).

    Answer: $\overline{Q} = c_1 V = 3$ cubic meters.   □

(c) Solve the differential equation in part (a) under the assumption that the there is no carbon monoxide in the room initially, and sketch the solution.

    Solution: The general solution of equation (2) is

$$Q(t) = \overline{Q} + Ce^{-\frac{F}{V}\,t},$$

where C is an arbitrary constant. If $Q(0) = 0$, then
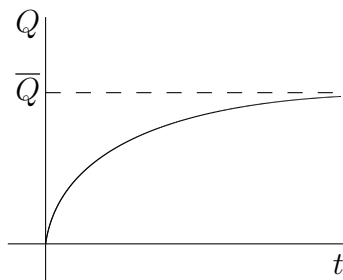
$$\overline{Q} + C = 0,$$

from which we get that $C = -\overline{Q}$. It then follows that

$$Q(t) = \overline{Q}(1 - e^{-\frac{F}{V}\,t}),$$

or

$$Q(t) = 3(1 - e^{-\frac{F}{V}\,t}). \tag{3}$$

in cubic meters. A sketch of the graph of $Q = Q(t)$ is shown in Figure 2.   □

Figure 2: Sketch of graph of $Q(t)$

(d) Based on your solution to part (c), give the concentration, $c(t)$, of carbon monoxide in the room (in percent volume) at any time t in minutes. What happens to the value of $c(t)$ in the long run?

Solution: Using the formula $c(t) = \dfrac{Q(t)}{V}$, we obtain from (3) that

$$c(t) = (0.05)(1 - e^{-\frac{F}{V}\,t}) \quad \text{for all } t \geqslant 0.$$

Since $\dfrac{F}{V} > 0$,

$$\lim_{t \to \infty} c(t) = 0.05,$$

or 5%. □

(e) Medical texts warn that exposure to air containing 0.1% carbon monoxide for some time can lead to a coma. How many hours does it take for the concentration of carbon monoxide found in part (d) to reach this level?

Solution: We want to find the time, t, for which $c(t) = 0.001$, or

$$(0.05)(1 - e^{-\frac{F}{V}\,t}) = 0.001,$$

or

$$1 - e^{-\frac{F}{V}\,t} = \frac{1}{50}.$$

Solving for t we obtain that

$$t = \frac{V}{F} \ln\left(\frac{50}{49}\right).$$

Substituting the values for V and F we obtain

$$t = 3 \times 10^4 \ln\left(\frac{50}{49}\right) \approx 606 \text{ minutes.}$$

Thus, it takes about 10.1 hours for the concentration of carbon monoxide in the room to reach the "coma" level.   □

5. **Newton's Law of Cooling.** The temperature in a hot iron decreases at a rate 0.11 times the difference between its present temperature and room temperature (20° C). Write a differential equation for the temperature of the iron. If the initial temperature of the rod is 100° C, and the time is measured in minutes, how long will it take for the rod to reach a temperature of 25° C?

> **Solution:** Let $u = u(t)$ denote the temperature in the rod at time t (we are assuming here that the temperature is the same through-out the rod at any time t; that is, the temperature is uniformly distributed). It R denotes the room temperature, then the differential equation describing the rate of change of u is
>
> $$\frac{du}{dt} = -k(u - R),$$
>
> where $k = 0.11$ and $R = 20°$ C.
>
> The general solution to the equation is then
>
> $$u(t) = R + ce^{-kt},$$
>
> where c is an arbitrary constant. Since the initial temperature is 100° C, we can compute c from the equation
>
> $$R + c = 100,$$
>
> from which we get that $c = 80°$ C. We then have that
>
> $$u(t) = 20 + 80e^{-kt},$$
>
> in centigrade degrees.
>
> Next, we find the time, t, for which $u(t) = 25$ degrees Celsius; that is, we solve for t in the equation
>
> $$20 + 80e^{-kt} = 25,$$
>
> or
>
> $$e^{-kt} = \frac{1}{16}.$$
>
> Taking the natural logarithm on both sides yields
>
> $$t = \frac{1}{k}\ln(16) \approx 25.2 \text{ minutes.}$$
>
>   □