Solutions to Assignment #9

1. Given a discrete random variable $X$ with a finite number of possible values $x_1, x_2, x_3, \ldots, x_N,$

the expected value of $X$ is defined to be the sum

$$E(X) = \sum_{i=1}^{N} x_i P[X = x_i].$$

Use this formula to compute the expected value of the numbers appearing on the top face of a fair die. Explain the meaning of this number.

Solution: Since $P[X = i] = \frac{1}{6}$ for $i = 1, 2, 3, 4, 5, 6$, it follows that

$$E(X) = \sum_{i=1}^{6} i \cdot \frac{1}{6} = \frac{1}{6} \sum_{i=1}^{6} i = \frac{1}{6} \cdot \frac{(6)(7)}{2} = \frac{7}{2}.$$

Thus, if we roll a die $n$ times, add up the outcomes, and divide by $n$, the result will be close to 3.5. \qed

2. Consider the following random experiment: Assume you have a fair die and you toss it until you get a six on the top face, and then you stop. Let $X$ denote the number of tosses you make until you stop.

(a) Explain why $X$ is a discrete random variable. What are the possible values for $X$?

Solution: Each time we repeat the experiment, the number of times it takes to get a “6” might differ from what it took the previous time. \qed

(b) For each value $x$ of $X$, compute $P[X = x]$; this is called the probability mass function, or pmf, of the random variable $X$.

Solution: The possible values of $X$ are 1, 2, 3, …, and the pmf is

$$P[X = n] = \left(\frac{5}{6}\right)^{n-1} \cdot \frac{1}{6} \quad \text{for} \quad n = 1, 2, 3, \ldots \quad \Box$$
3. Given a discrete random variable $X$ with an infinite number of possible values $x_1, x_2, x_3, \ldots$

the expected value of $X$ is defined to be the infinite series

$$E(X) = \sum_{i=1}^{\infty} x_i P[X = x_i].$$

Use this formula to compute the expected value random variable $X$ of the previous problem; that is, $X$ is the number of times you need to toss a fair die until you get a six on the top face.

**Solution:** In order to do this problem, first we consider the general situation in which an experiment consists of repeated independent trials until a specified outcome of probability $p$, with $0 < p < 1$, occurs. We assume that each trial has two possible outcomes: the one with probability $p$, and the other with probability $1 - p$. In the case of the fair die, one outcome is to get a six with $p = \frac{1}{6}$, and the other is the outcome of not getting a six. In the general case, the pmf is given by

$$P[X = n] = (1 - p)^{n-1} \cdot p \quad \text{for } n = 1, 2, 3, \ldots$$

Thus,

$$E(X) = \sum_{n=1}^{\infty} n \cdot P[X = n]$$

$$= \sum_{n=1}^{\infty} n \cdot (1 - p)^{n-1} \cdot p$$

$$= p \sum_{n=1}^{\infty} n(1 - p)^{n-1}.$$

Observe that $n(1 - p)^{n-1}$ is the derivative with respect to $p$ of $-(1 - p)^n$. It then follows that

$$E(X) = -p \sum_{n=1}^{\infty} \frac{d}{dp} [(1 - p)^n]$$

$$= -p \left\{ \sum_{n=1}^{\infty} (1 - p)^n \right\}$$

$$= -p \left\{ \frac{1 - p}{1 - (1 - p)} \right\} \quad \text{since } 0 < 1 - p < 1,$$
where we have added up the convergent geometric series \( \sum_{n=1}^{\infty} (1 - p)^n \).

Simplifying we get

\[
E(X) = -p \frac{d}{dp} \left( \frac{1}{p} - 1 \right) = -p \cdot \left( -\frac{1}{p^2} \right) = \frac{1}{p}.
\]

Thus, for the case \( p = \frac{1}{6} \) we get that \( E(X) = 6 \). Hence, on average, it takes six tosses to get a six when rolling a fair die. \( \square \)

4. Let \( M(t) \) denote number of bacteria in a colony of initial size \( N_0 \) which develop mutations in the time interval \([0, t]\). It was shown in the lectures that if there are no mutations at time \( t = 0 \), and if \( M(t) \) follows the assumptions of a Poisson process, then the probability of no mutations in the time interval \([0, t]\) is given by

\[
P_0(t) = P[M(t) = 0] = e^{-\lambda t}
\]

where \( \lambda > 0 \) is the average number of mutations per unit time, or the mutation rate.

Let \( T > 0 \) denote the time at which the first mutation occurs.

(a) Explain why \( T \) is a random variable. Observe that it is a continuous random variable.

\textbf{Solution:} Suppose we start observing the bacterial population at time \( t = 0 \) when its size is \( N_0 \). If we can observe the first mutation, then \( T \) is the time of that observation. If we repeat the experiment, starting with the same number of bacteria \( N_0 \), and under the same conditions, then the value for \( T \) will most likely be different from the previously obtained one. Thus, \( T \) is a random variable. \( \square \)

(b) For any \( t > 0 \), explain why the statement

\[
P[T > t] = P[M(t) = 0]
\]

is true, and use it to compute

\[
F(t) = P[T \leq t].
\]
The function $F(t)$, usually denoted by $F_T(t)$, is called the cumulative distribution function, or cdf, of the random variable $T$.

**Solution:** If $T > t$, then no mutation has occurred at time $t$, and therefore the probability of that event is the same as the probability of the event $[M(t) = 0]$. Hence,

$$P[T > t] = P_0(t) = e^{-\lambda t}, \quad \text{for } t \geq 0$$

and so

$$F_T(t) = P[T \leq t] = 1 - P[T > t] = 1 - e^{-\lambda t}$$

for $t \geq 0$. On the other, if $t < 0$ then $P[T > t] = P[T > 0] = 1$, since $T$ is nonnegative. It then follows that for $t < 0$,

$$P[T \leq t] = 1 - P[T > t] = 1 - 1 = 0$$

and therefore

$$F_T(t) = \begin{cases} 
0 & \text{if } t < 0 \\
1 - e^{-\lambda t} & \text{if } t \geq 0.
\end{cases}$$

(c) Compute the derivative $f(t) = F'(t)$ of the cdf $F$ obtained in the previous part.

The function $f(t)$, usually denoted by $f_T(t)$, is called the probability density function, or pdf, of the random variable $T$.

**Solution:** First, observe that $f_T(t) = \frac{d}{dt}(1 - e^{-\lambda t}) = \lambda e^{-\lambda t}$ for $t > 0$. The function $F_T$ is not differentiable at 0. However, we can define

$$F_T(t) = \begin{cases} 
0 & \text{if } t \leq 0 \\
\lambda e^{-\lambda t} & \text{if } t > 0,
\end{cases}$$

and still get a valid pdf. □

5. Given a continuous random variable $X$ with pdf $f_X$, the expected value of $X$ is defined to be

$$E(X) = \int_{-\infty}^{\infty} x f_X(x) \, dx.$$ 

Use this formula to compute the expected value of the $T$, where $T$ is the random variable defined in the previous problem; that is, $T > 0$ is he time at which the first mutation occurs for a bacterial colony exposed to a virus at time $t = 0$, assuming that there are no mutations at that time. How does this value relate to the average mutation rate $\lambda$?
Solution: \(E(T) = \int_{-\infty}^{\infty} t f_T(t) \, dt = \int_{0}^{\infty} t \lambda e^{-\lambda t} \, dt\). Integrating by parts we get

\[
E(T) = -te^{-\lambda t}\bigg|_{0}^{\infty} + \int_{0}^{\infty} e^{-\lambda t} \, dt
\]

\[
= 0 + \left[ -\frac{1}{\lambda} e^{-\lambda t} \right]_{0}^{\infty}
\]

\[
= \frac{1}{\lambda}.
\]

Thus, the expected value of \(T\) is the reciprocal of \(\lambda\). \(\Box\)