Solutions to Assignment #9

1. Given a discrete random variable X with a finite number of possible values

$$x_1, x_2, x_3, \ldots, x_N,$$

the expected value of X is defined to be the sum

$$E(X) = \sum_{i=1}^{N} x_i P[X = x_i].$$

Use this formula to compute the expected value of the numbers appearing on the top face of a fair die. Explain the meaning of this number.

Solution: Since $P[X = i] = \dfrac{1}{6}$ for $i = 1, 2, 3, 4, 5, 6$, it follows that

$$E(X) = \sum_{i=1}^{6} i \frac{1}{6} = \frac{1}{6} \sum_{i=1}^{6} i = \frac{1}{6} \frac{(6)(7)}{2} = \frac{7}{2}.$$

Thus, if we roll a die n times, add up the outcomes, and divide by n, the result will be close to 3.5. □

2. Consider the following random experiment: Assume you have a fair die and you toss it until you get a six on the top face, and then you stop. Let X denote the number of tosses you make until you stop.

   (a) Explain why X is a discrete random variable. What are the possible value for X?

   Solution: Each time we repeat the experiment, the number of times it takes to get a "6" might differ from what it took the previous time. □

   (b) For each value x of X, compute $P[X = x]$; this is called the *probability mass function,* or pmf, of the random variable X.

   Solution: The possible values of X are $1, 2, 3, \ldots$, and the pmf is

$$P[X = n] = \left(\frac{5}{6}\right)^{n-1} \cdot \frac{1}{6} \quad \text{for } \; n = 1, 2, 3, \ldots \qquad \square$$

3. Given a discrete random variable X with an infinite number of possible values

$$x_1, x_2, x_3, \ldots$$

the expected value of X is defined to be the infinite series

$$E(X) = \sum_{i=1}^{\infty} x_i P[X = x_i].$$

Use this formula to compute the expected value random variable X of the previous problem; that is, X is the number of times you need to toss a fair die until you get a six on the top face.

Solution: In order to do this problem, first we consider the general situation in which an experiment consists of repeated independent trials until a specified outcome of probability p, with $0 < p < 1$, occurs. We assume that each trial has two possible outcomes: the one with probability p, and the other with probability $1 - p$. In the case of the fair die, one outcome is to get a six with $p = \dfrac{1}{6}$, and the other is the outcome of not getting a six. In the general case, the pmf is given by

$$P[X = n] = (1 - p)^{n-1} \cdot p \quad \text{for} \quad n = 1, 2, 3, \ldots$$

Thus,

$$\begin{aligned} E(X) &= \sum_{n=1}^{\infty} n \cdot P[X = n] \\ &= \sum_{n=1}^{\infty} n \cdot (1 - p)^{n-1} \cdot p \\ &= p \sum_{n=1}^{\infty} n(1 - p)^{n-1}. \end{aligned}$$

Observe that $n(1 - p)^{n-1}$ is the derivative with respect to p of $-(1 - p)^n$. It then follows that

$$\begin{aligned} E(X) &= -p \sum_{n=1}^{\infty} \frac{d}{dp}[(1 - p)^n] \\ &= -p \, \frac{d}{dp} \left[\sum_{n=1}^{\infty} (1 - p)^n \right] \\ &= -p \, \frac{d}{dp} \left(\frac{1 - p}{1 - (1 - p)} \right) \qquad \text{since } 0 < 1 - p < 1, \end{aligned}$$

where we have added up the convergent geometric series $\sum_{n=1}^{\infty}(1-p)^n$.

Simplifying we get

$$\begin{aligned} E(X) &= -p\,\frac{d}{dp}\left(\frac{1}{p}-1\right) \\ &= -p\cdot\left(-\frac{1}{p^2}\right) \\ &= \frac{1}{p}. \end{aligned}$$

Thus, for the case $p = \frac{1}{6}$ we get that $E(X) = 6$. Hence, on average, it takes six tosses to get a six when rolling a fair die. □

4. Let $M(t)$ denote number of bacteria in a colony of initial size N_o which develop mutations in the time interval $[0, t]$. It was shown in the lectures that if there are no mutations at time $t = 0$, and if $M(t)$ follows the assumptions of a Poisson process, then the probability of no mutations in the time interval $[0, t]$ is given by

$$P_0(t) = P[M(t) = 0] = e^{-\lambda t}$$

where $\lambda > 0$ is the average number of mutations per unit time, or the *mutation rate.*

Let $T > 0$ denote the time at which the first mutation occurs.

(a) Explain why T is a random variable. Observe that it is a *continuous* random variable.

Solution: Suppose we start observing the bacterial population at time $t = 0$ when its size is N_o. If we can observe the first mutation, then T is the time of that observation. If we repeat the experiment, starting with the same number of bacteria N_o, and under the same conditions, then the value for T will most likely be different from the previously obtained one. Thus, T is a random variable. □

(b) For any $t > 0$, explain why the statement

$$P[T > t] = P[M(t) = 0]$$

is true, and use it to compute

$$F(t) = P[T \leq t].$$

The function $F(t)$, usually denoted by $F_T(t)$, is called the *cumulative distribution function,* or cdf, of the random variable T.

Solution: It $T > t$, then no mutation has occurred at time t, and therefore the probability of that event is the same as the probability of the event $[M(t) = 0]$. Hence,

$$P[T > t] = P_0(t) = e^{-\lambda t}, \quad \text{for } t \geq 0$$

and so

$$F_T(t) = P[T \leq t] = 1 - P[T > t] = 1 - e^{-\lambda t}$$

for $t \geq 0$. On the other, if $t < 0$ then $P[T > t] = P[T > 0] = 1$, since T is nonnegative. It then follows that for $t < 0$,

$$P[T \leq t] = 1 - P[T > t] = 1 - 1 = 0$$

and therefore

$$F_T(t) = \begin{cases} 0 & \text{if } t < 0 \\ 1 - e^{-\lambda t} & \text{if } t \geq 0. \end{cases}$$

(c) Compute the derivative $f(t) = F'(t)$ of the cdf F obtained in the previous part.

The function $f(t)$, usually denoted by $f_T(t)$, is called the *probability density function,* or pdf, of the random variable T.

Solution: First, observe that $f_T(t) = \dfrac{d}{dt}(1 - e^{-\lambda t}) = \lambda e^{-\lambda t}$ for $t > 0$. The function F_T is not differentiable at 0. However, we can define

$$F_T(t) = \begin{cases} 0 & \text{if } t \leq 0 \\ \lambda e^{-\lambda t} & \text{if } t > 0, \end{cases}$$

and still get a valid pdf. □

5. Given a continuous random variable X with pdf f_X, the *expected value* of X is defined to be

$$E(X) = \int_{-\infty}^{\infty} x f_X(x) dx.$$

Use this formula to compute the expected value of the T, where T is the random variable defined in the previous problem; that is, $T > 0$ is he time at which the first mutation occurs for a bacterial colony exposed to a virus at time $t = 0$, assuming that there are no mutations at that time. How does this value relate to the average mutation rate λ?

Solution: $E(T) = \int_{-\infty}^{\infty} t f_T(t) \, \mathrm{d}t = \int_0^{\infty} t \, \lambda e^{-\lambda t} \, \mathrm{d}t$. Integrating by parts we get

$$\begin{aligned} E(T) &= -te^{-\lambda t}\Big|_0^{\infty} + \int_0^{\infty} e^{-\lambda t} \, \mathrm{d}t \\ &= 0 + \left[-\frac{1}{\lambda} e^{-\lambda t}\right]_0^{\infty} \\ &= \frac{1}{\lambda}. \end{aligned}$$

Thus, the expected value of T is the reciprocal of λ. □